

# Complete Elgot Monads and Coalgebraic Resumptions

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**Abstract.** *Monads* are extensively used nowadays to abstractly model a wide range of computational effects such as nondeterminism, statefulness, and exceptions. It turns out that equipping a monad with a (uniform) iteration operator satisfying a set of natural axioms allows for modelling iterative computations just as abstractly. The emerging monads are called *complete Elgot monads*. It has been shown recently that extending complete Elgot monads with free effects (e.g. operations of sending/receiving messages over channels) canonically leads to *generalized coalgebraic resumption monads*, previously used as semantic domains for non-wellfounded guarded processes. In this paper, we continue the study of the relationship between abstract complete Elgot monads and those that capture coalgebraic resumptions, by comparing the corresponding categories of (Eilenberg-Moore) algebras. To this end we first provide a characterization of the latter category; even more generally, we formulate this characterization in terms of Uustalu’s parametrized monads. This is further used for establishing a characterization of complete Elgot monads as precisely those monads whose algebras are coherently equipped with the structure of algebras of coalgebraic resumption monads.

## 1 Introduction

One traditional use of monads in computer science, stemming from the seminal thesis of Lawvere [18], is as a tool for algebraic semantics where monads arise as a high-level metaphor for (clones of) equational theories. More recently, Moggi proposed to associate monads with *computational effects* and use them as a generic tool for denotational semantics [20], which later had a considerable impact on the design of functional programming languages, most prominently Haskell [1]. Finally, in the first decade of the new millennium, Plotkin and Power reestablished the connection between computational monads and algebraic theories in their theory of *algebraic effects* [21, 22].

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We use the outlined perspective to study the notion of *iteration*, a concept, that has a well-established algebraic description, and whose relevance in the context of computational effects is certain. On the technically level our present work can be viewed as a continuation of the previous extensive work on monads with iteration [2, 5, 8] having its roots in the work of Elgot [12] and Bloom and Ésik [11] on iteration theories.

More specifically, we are concerned with a particular construction on monads: given a monad  $\mathbb{T}$  and a functor  $\Sigma$ , we assume the existence of the coalgebra

$$T_\Sigma X = \nu\gamma. T(X + \Sigma\gamma) \quad (\star)$$

for each object  $X$  (these final coalgebras exist under mild assumptions on  $T$ ,  $\Sigma$ , and the base category). It is known [26] that  $T_\Sigma$  extends to a monad  $\mathbb{T}_\Sigma$  and we call the latter the *generalized coalgebraic resumption monad*.

Intuitively,  $(\star)$  is a generic semantic domain for systems combining *extensional* (via  $\mathbb{T}$ ) and *intensional* (via  $\Sigma$ ) features with iteration. To make this intuition more precise, consider the following simplistic

**Example 1.** Let  $A = \{a, b\}$  be an alphabet of *actions*. Then the following system of equations specifies *processes*  $x_1, x_2, x_3$  of *basic process algebra (BPA)*:

$$x_1 = a \cdot (x_2 + x_3) \quad x_2 = a \cdot x_1 + b \cdot x_3 \quad x_3 = a \cdot x_1 + \checkmark$$

We can think of this specification as a map  $P \rightarrow T(\{\checkmark\} + \Sigma P)$  where  $P = \{x_1, x_2, x_3\}$ ,  $\Sigma = A \times -$  and  $T = \mathcal{P}_\omega$  is the finite powerset monad. Using the standard approach [24] we can *solve* this specification by finding a map  $P \rightarrow T_\Sigma\{\checkmark\}$  that assigns to every  $x_i$  the corresponding semantics over the domain of possibly non-wellfounded trees  $T_\Sigma\{\checkmark\} = \nu\gamma. \mathcal{P}_\omega(\{\checkmark\} + A \times \gamma)$ . The crucial fact here is that the original system is *guarded*, i.e. every recursive call of a variable  $x_i$  is preceded by an action. In particular, this implies that the recursive system at hand has a unique solution.

If the guardedness assumption is dropped, the uniqueness of solutions can no longer be ensured, but it is possible to introduce a notion of *canonical solution* assuming that the monad  $\mathbb{T}$  has suitable completeness properties under an order, or more generally is a *complete Elgot monad*. A monad  $\mathbb{T}$  is called a complete Elgot monad if it defines a *solution*  $f^\dagger : X \rightarrow TY$  for every morphisms of the form  $f : X \rightarrow T(Y + X)$  satisfying a certain well-established set of axioms for iteration (e.g.  $\mathcal{P}_\omega$  is not a complete Elgot monad, but the countable powerset monad  $\mathcal{P}_{\omega_1}$  is). The central result of the recent work [14] is that whenever  $\mathbb{T}$  is a complete Elgot monad then so is the transformed monad  $(\star)$ . In particular, this allows for solving recursive equations over processes (in the sense of Example 1) whenever recursive equations over  $\mathbb{T}$  are solvable.

In the present paper we study the relationship between guarded and unguarded recursion implemented via complete Elgot monads and generalized coalgebraic resumptions, respectively. As an auxiliary abstraction device, we involve the notion of *parametrized monad* previously developed by Uustalu [26], e.g. the bifunctor  $X \# Y = T(X + \Sigma Y)$  in  $(\star)$  is a parametrized monad.

The paper is organized as follows. After categorical preliminaries (Section 2) we present and discuss complete Elgot monads in Section 3. In Section 4 we introduce algebras and complete Elgot algebras for parametrized monads; here we show that existence of free complete Elgot algebras is equivalent to the existence of certain final coalgebras, which then form carriers of the corresponding algebras (Theorem 20); furthermore, we show that the category of complete Elgot algebras is equivalent to the Eilenberg-Moore category of a generalized coalgebraic resumption monad over the corresponding parametrized monad (Theorem 27). Finally, in Section 5 we apply the developed results to characterize complete Elgot monads as those whose algebras are coherently equipped with complete Elgot algebra structures (Theorem 32 and 33).

## 2 Preliminaries

We assume that readers are familiar with basic category theory [19]; we write  $|\mathbf{C}|$  for the class of objects of a category  $\mathbf{C}$  and  $f : X \rightarrow Y$  for morphisms in  $\mathbf{C}$ . We often omit indexes, e.g. on natural transformations, if they are clear from the context.

In this paper we work with an ambient category  $\mathbf{C}$  with finite coproducts. We denote by  $\text{inl}$  and  $\text{inr}$  the left- and right-hand coproduct injections from  $X$  and  $Y$  to  $X + Y$ , and  $[f, g] : X + Y \rightarrow Z$  the is the *copair* of  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$ , i.e. the unique morphism with  $[f, g] \text{inl} = f$  and  $[f, g] \text{inr} = g$ . The codiagonal is denoted by  $\nabla = [\text{id}, \text{id}] : X + X \rightarrow X$  as usual.

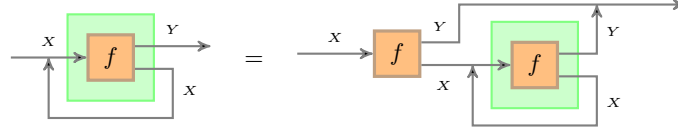
We consider *monads* by  $\mathbf{C}$  given in the form of *Kleisli triples*  $\mathbb{T} = (T, \eta, -^*)$  where  $T$  is an endomap on  $|\mathbf{C}|$ ,  $\eta$ , called *monad unit*, is a family of morphisms  $\eta_X : X \rightarrow TX$  indexed over  $|\mathbf{C}|$ , and (*Kleisli*) *lifting* assigning to each  $f : X \rightarrow TY$  a morphism  $f^* : TX \rightarrow TY$  such that the following laws hold:

$$\eta^* = \text{id}, \quad f^* \eta = f, \quad (f^* g)^* = f^* g^*.$$

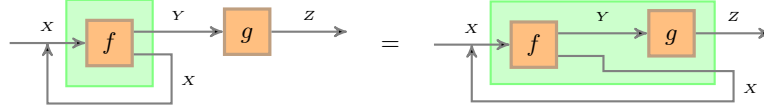
This is equivalent to the definition of a monad in terms of *monad multiplication*  $\mu$  [19], where in particular  $\mu = \text{id}^*$ ,  $\eta$  extends to a natural transformation, and  $T$  to an endofunctor by  $Tf = (\eta f)^*$ . The *Kleisli category*  $\mathbf{C}_{\mathbb{T}}$  of  $\mathbb{T}$  is formed by *Kleisli morphisms*:  $\text{Hom}_{\mathbf{C}_{\mathbb{T}}}(X, Y) = \text{Hom}_{\mathbf{C}}(X, TY)$  under  $\eta$  used as identity morphisms and *Kleisli composition*  $(f, g) \mapsto f \diamond g = f^* g$ . We adopt the notation  $f : X \multimap Y$  for Kleisli morphisms  $f : X \rightarrow TY$ .

The forgetful functor from  $\mathbf{C}_{\mathbb{T}}$  to  $\mathbf{C}$  has a left adjoint sending any  $f : X \rightarrow Y$  to  $\underline{f} = \eta f : X \rightarrow TY$ . Like any left adjoint, this functor preserves colimits, and in particular coproducts. Since  $|\mathbf{C}| = |\mathbf{C}_{\mathbb{T}}|$ , this implies that coproducts in  $\mathbf{C}_{\mathbb{T}}$  exist and are lifted from  $\mathbf{C}$ . Explicitly,  $\underline{\text{inl}} = \eta \text{inl} : X \multimap X + Y$ ,  $\underline{\text{inr}} = \eta \text{inr} : X \multimap X + Y$  are the coproduct injections in  $\mathbf{C}_{\mathbb{T}}$  and  $[f, g] : A + B \multimap C$  is the copair of  $A \multimap C$  and  $B \multimap C$ . We denote by  $f \oplus g : A + B \multimap A' + B'$  the coproduct of morphisms  $f : A \multimap A'$  and  $g : B \multimap B'$  in  $\mathbf{C}_{\mathbb{T}}$ . Besides  $\mathbf{C}_{\mathbb{T}}$ , we consider the category  $\mathbf{C}^{\mathbb{T}}$  of (*Eilenberg-Moore*) *algebras* for  $\mathbb{T}$ , whose objects are pairs  $(A, a : TA \rightarrow A)$ , satisfying laws:  $a \eta = \text{id}$ ,  $a(Ta) = a \mu$ ; see [19] for more details.

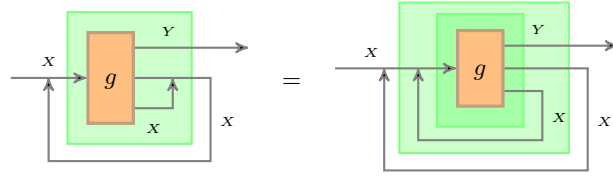
Fixpoint:



Naturality:



Codiagonal:



Uniformity:

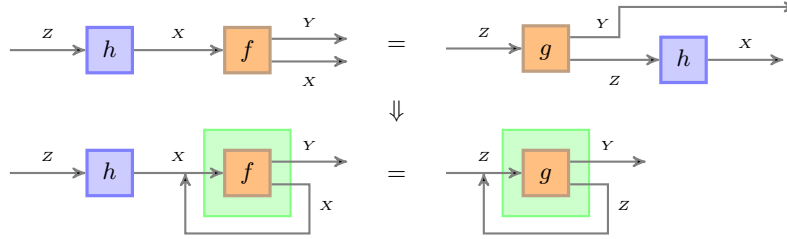


Fig. 1: Axioms of complete Elgot monads.

We call on the standard facts on  $(F\text{-})\text{coalgebras}$  [23], which are pairs of the form  $(X, f)$  with *carriers*  $X$  ranging over  $|\mathbf{C}|$  and *transition structures*  $f$  ranging over  $\text{Hom}_{\mathbf{C}}(X, FX)$  for a fixed endofunctor  $F$ . Coalgebras together with morphisms of the carriers compatible with the transition structure form a category. The final  $F$ -coalgebra, if it exists, is denoted  $(\nu F, \text{out})$ . By Lambek's lemma,  $\text{out}$  is an isomorphism, whose inverse  $\text{out}^{-1}$  can be obtained as  $\text{coit}(F \text{out})$  where for any coalgebra  $(X, f : X \rightarrow FX)$  we denote by  $\text{coit } f$  the unique coalgebra morphism  $X \rightarrow \nu F$  to the final coalgebra.

### 3 Complete Elgot Monads for Iteration

Complete Elgot monads are a slight generalization of Elgot monads from [8, 9], which in turn, for the base category being  $\mathbf{Set}$ , correspond precisely to those iteration theories of Bloom and Ésik [11] that satisfy the functorial dagger implication for base morphisms. In the following definition cited from [14], we follow

the terminology of [10, 25] where the same axioms were considered in the dual setting of generic parametrized recursion.

**Definition 2 (Complete Elgot monads).** A *complete Elgot monad* is a monad  $\mathbb{T}$  equipped with an operator  $-^\dagger$ , called *iteration*, that assigns to each morphism  $f : X \multimap Y + X$  a morphism  $f^\dagger : X \multimap Y$  such that the following axioms hold:

- *fixpoint*:  $f^\dagger = [\eta, f^\dagger] \diamond f$ , for  $f : X \multimap Y + X$ ;
- *naturality*:  $g \diamond f^\dagger = ((g \oplus \eta) \diamond f)^\dagger$  for  $g : Y \multimap Z$ ;
- *codiagonal*<sup>1</sup>:  $([\eta, \text{inr}] \diamond g)^\dagger = (g^\dagger)^\dagger$  for  $g : X \multimap (Y + X) + X$ ;
- *uniformity*:  $f \diamond \underline{h} = (\eta \oplus \underline{h}) \diamond g$  implies  $f^\dagger \diamond \underline{h} = g^\dagger$  for  $g : Z \multimap Y + Z$  and  $h : Z \rightarrow X$ .

The above axioms of iteration can be comprehensibly represented in a flowchart-style as in Fig. 1. Here the feedback loops correspond to iteration and the colored frames indicate the scope of the constructs being iterated. We believe that this presentation illustrates that these axioms are natural and desirable laws of iteration. For example, the naturality axiom expresses the fact that the scope of the iteration can be stretched to embrace a function postprocessing the output of the terminating branch. There is an obvious similarity between the axioms in Fig. 1 and the axioms of *traced monoidal categories* [17]. In fact, Hasegawa [16] proved that there is an equivalent presentation of a dagger operation satisfying the above axioms in terms of a uniform trace operator w.r.t. coproducts (actually, Hasegawa worked in the dual setting with products). Note that the present axioms make use of coproduct injections and the codiagonal morphism, while the trace axioms can be formulated more generally for any monoidal product.

One standard source of examples for complete Elgot monads is a suitable enrichment of the Kleisli category  $\mathbf{C}_\mathbb{T}$  over complete partial orders.

**Example 3. ( $\omega$ -continuous monads)** An  $\omega$ -continuous monad consists of a monad  $\mathbb{T}$  such that the Kleisli category  $\mathbf{C}_\mathbb{T}$  is enriched over the category  $\mathbf{Cppo}$  of  $\omega$ -complete partial orders with bottom  $\perp$  and (nonstrict) continuous maps; moreover, composition in  $\mathbf{C}$  is required to be left strict and composition in  $\mathbf{C}_\mathbb{T}$  right strict:  $\perp f = \perp$ ,  $f \diamond \perp = \perp$ ; equivalently,  $\perp$  is a *constant* of  $\mathbb{T}$ . Note that it follows that copairing in  $\mathbf{C}_\mathbb{T}$  is continuous in both arguments; for  $\bigsqcup_i [f_i, g]$  is a morphism satisfying  $(\bigsqcup_i [f_i, g]) \text{inl} = \bigsqcup_i f_i$  and  $(\bigsqcup_i [f_i, g]) \text{inr} = g$  by the continuity of composition, whence  $\bigsqcup_i [f_i, g] = [\bigsqcup_i f_i, g]$  (and similarly for continuity in the second argument).

It is shown in [14] that an  $\omega$ -continuous monad is a complete Elgot monad with  $e^\dagger$  calculated as the least fixed point of the map  $f \mapsto [\eta, f] \diamond e$ . This yields the powerset monad  $\mathcal{P}$ , the *Maybe-monad*  $(- + 1)$ , or the nondeterministic state monad  $\mathcal{P}(- \times S)^S$  as examples of complete Elgot monads on **Set**. The *lifting monad*  $(-)_\perp$  is a complete Elgot monad on the category of complete partial orders without bottom.

<sup>1</sup> The codiagonal axiom is often written as  $((\eta \oplus \nabla) \diamond g)^\dagger = (g^\dagger)^\dagger$  implicitly alluding to the canonical isomorphism  $Y + (X + X) \cong (Y + X) + X$ .

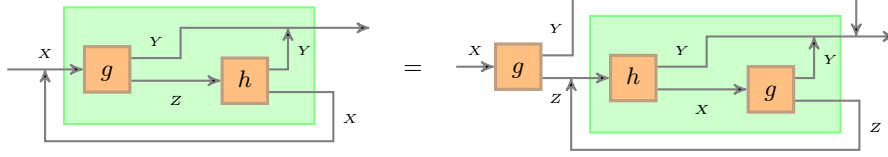


Fig. 2: Dinaturality axiom.

Another principal source of examples are free complete Elgot monads for which the iteration of guarded morphisms is uniquely defined.

**Example 4. (Free complete Elgot monads)** Suppose  $\mathbb{T}$  is the initial complete Elgot monad. It is shown in [14] that whenever the functor  $\mathbb{T}_\Sigma$  defined by  $(\star)$  exists, it yields the *free complete Elgot monad on  $\Sigma$*  (note that the original  $\mathbb{T}$  is the free complete Elgot monads on  $\Sigma$  being the constant functor on the initial object of  $\mathbf{C}$ ). On **Set** (more generally, on any *hyperextensive category* [3]) the initial complete Elgot monad  $\mathbb{T}$  is the Maybe-monad  $- + 1$ .

In comparison to the previous work [14], Definition 2 remarkably drops the axiom of *dinaturality* (see Fig. 2). The reason for it is that this axiom turns out to be derivable, which is a fact that was recently discovered and formalized on the level of abstract iteration theories [13]. Corollary 6 from *op.cit.* can be couched in present terms (modulo the terminological change: *parameter identity* instead of *naturality*, *double dagger* instead of *codiagonal* and *dagger implication for base morphisms* instead of *uniformity*) as follows:

**Proposition 5 (Dinaturality).** *Given  $g : X \multimap Y + Z$  and  $h : Z \multimap Y + X$ , then*

$$([\text{inl}, h] \diamond g)^\dagger = [\eta, ([\text{inl}, g] \diamond h)^\dagger] \diamond g$$

The codiagonal axiom in Definition 2 can equivalently be replaced by a form of the well-known *Bekić identity*, see [11].

**Proposition 6 (Bekić identity).** *A complete Elgot monad  $\mathbb{T}$  is, equivalently, a monad satisfying the fixpoint, naturality and uniformity axioms (as in Definition 2), and the Bekić identity*

$$(T\alpha[f, g])^\dagger = [\eta, h^\dagger] \diamond [\text{inr}, g^\dagger]$$

where  $g : X \multimap (Z + Y) + X$ ,  $f : Y \multimap (Z + Y) + X$ ,  $h = [\eta, g^\dagger] \diamond f : Y \multimap Z + Y$ , with  $\alpha : (A + B) + C \rightarrow A + (B + C)$  being the obvious associativity morphism.

## 4 Parametrized Monads and Complete Elgot Algebras

In order to study complete Elgot monads and their algebras it is helpful to make a further abstraction step and generalize from monads to *parametrized monads* [26] (finitary parametrized monads are also called *bases* [4]), which are of independent interest.

**Definition 7. (Parameterized monad)** A *parameterized monad* over  $\mathbf{C}$  is a functor from  $\mathbf{C}$  to the category of monads over  $\mathbf{C}$  and monad morphisms. More explicitly, a parameterized monad is a bifunctor  $\# : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$  such that for any  $X \in |\mathbf{C}|$ ,  $- \# X : \mathbf{C} \rightarrow \mathbf{C}$  is a monad, and for any  $f : X \rightarrow Y$ , the family  $(\text{id}_Z \# f)_Z$  yields a monad morphism from  $- \# X$  to  $- \# Y$ .

*Remark 8.* The order of arguments in  $X \# Y$  is in agreement with [26] and differs from [4] where the notation  $Y \square X$  equivalent to the present  $X \# Y$  is used. We chose the order of arguments to ensure agreement with the type profile of the iteration operator  $-^\dagger$ , which is in turn in agreement with the expression  $(\star)$ .

Following [4] we will from now on denote the unit and monad multiplication of monads  $- \# X$  by  $u_A^X : A \rightarrow A \# X$  and  $m_A^X : (A \# X) \# X \rightarrow A \# X$ , respectively.

**Example 9. (Parametrized monads)** We recall some standard examples of parametrized monads from [26]; further examples can be found e.g. in [7].

1. Whenever  $\mathbb{T} = (T, \eta, -^*)$  is a monad and  $\Sigma$  is a functor,  $A \# X = T(A + \Sigma X)$  is a parametrized monad with the unit given by

$$u_A^X = \left( A \xrightarrow{\text{inl}} A + \Sigma X \xrightarrow{\eta_{A + \Sigma X}} T(A + \Sigma X) \right)$$

and the multiplication by

$$m_A^X = \left( T(T(A + \Sigma X) + \Sigma X) \xrightarrow{[\text{id}, \eta_{A + \Sigma X} \text{ inr}]^*} T(A + \Sigma X) \right).$$

Specifically, if  $\Sigma$  is the constant functor on object  $E$  then  $X \# Y$  is the exception monad transformer with exceptions from  $E$  [20]. Another interesting special case is when  $\mathbb{T}$  is the identity monad (cf. Remark 15).

2.  $A \# X = A \times X^*$  is a parametrized monad with the unit and multiplication given by

$$u_A^X : a \mapsto (a, \varepsilon) \quad \text{and} \quad m_A^X : (a, w, v) \mapsto (a, wv),$$

where  $\varepsilon$  denotes the empty word and  $wv$  concatenation of words.

3. Given a contravariant endfunctor  $H$ ,  $A \# X = A^{HX}$  is a parametrized monad with the unit and multiplication given by

$$u_A^X : a \mapsto \lambda x. a \quad \text{and} \quad m_A^X : (f : HX \rightarrow (HX \rightarrow A)) \mapsto \lambda x. f(x)(x).$$

This is a generalization of the well known *reader monad*, which can be recovered by instantiating  $H$  with a constant functor.

The following is a straightforward extension of the notion of an algebra for a base studied in [4] to arbitrary parametrized monads.

**Definition 10. (#-algebras)** Given a parameterized monad  $\# : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ , a #-algebra is a pair  $(A, a)$  consisting of an object  $A$  of  $\mathbf{C}$ , and an algebra for the monad  $- \# A$ , i.e. a morphism  $a : A \# A \rightarrow A$  satisfying

$$\begin{array}{ccc} A & \xrightarrow{u_A^A} & A \# A \\ & \searrow \text{id} & \downarrow a \\ & & A \end{array} \quad \begin{array}{ccc} (A \# A) \# A & \xrightarrow{a \# \text{id}} & A \# A \\ m_A^A \downarrow & & \downarrow a \\ A \# A & \xrightarrow{a} & A \end{array}$$

A morphism between #-algebras  $(A, a)$  and  $(B, b)$  is a  $\mathbf{C}$ -morphism  $f : A \rightarrow B$  such that  $f a = b(f \# f)$ .

**Example 11.** Several examples of #-algebras have been discussed in [6, 7]. Here we recall from *loc. cit.* only the following. Consider the three bases  $A \#_1 X = A + X \times X$ ,  $A \#_2 X = A \times X^*$ , and  $A \#_3 X = BA$  on **Set** where  $BA$  is the free algebra with one binary operation on  $A$  (i.e.  $BA$  consists of all finite binary trees with leaves labelled in  $A$ ). Note that  $\#_1$  and  $\#_3$  are special cases of the parameterized monad of Example 9(i) for  $\mathbb{T}$  the identity monad and  $\Sigma X = X \times X$  and  $\Sigma X = \emptyset$ , respectively. The category of algebras is in each of the three cases isomorphic to the category of algebras with one binary operation. Later, when we discuss complete Elgot #-algebras, we are going to see a difference between these three parameterized monad.

For our leading example  $X \# Y = T(X + \Sigma Y)$  the category of #-algebras can be described explicitly.

**Proposition 12.** *Let  $X \# Y = T(X + \Sigma Y)$  for a monad  $\mathbb{T}$  and a functor  $\Sigma$  on  $\mathbf{C}$ . Then #-algebras are precisely  $\mathbb{T}$ - $\Sigma$ -bialgebras, i.e. triples  $(A, a, f)$  where  $a : TA \rightarrow A$  is a  $\mathbb{T}$ -algebra and  $f : \Sigma A \rightarrow A$  is a  $\Sigma$ -algebra.*

**Corollary 13.** *Let  $X \# Y = T(X + Y)$  for a monad  $\mathbb{T}$  on  $\mathbf{C}$ . The category  $\mathbf{C}^{\mathbb{T}}$  of  $\mathbb{T}$ -algebras is isomorphic to the full subcategory of those #-algebras  $a : T(A + A) \rightarrow A$ , which factor through  $T\nabla$ .*

Analogously to the case of monads, we introduce #-algebras with iteration. This generalizes the definition of a *complete Elgot algebra for a functor* from [5].

**Definition 14. (Complete Elgot #-algebras)** A complete Elgot #-algebra is a #-algebra  $a : A \# A \rightarrow A$  equipped with an iteration operator

$$\frac{e : X \rightarrow A \# X}{e^\dagger : X \rightarrow A}$$

satisfying the following axioms:

- *solution*: for every  $e : X \rightarrow A \# X$  we have  $e^\dagger = a(\text{id} \# e^\dagger) e$ ;
- *functoriality*: for every  $e : X \rightarrow A \# X$ ,  $f : Y \rightarrow A \# X$  and  $h : X \rightarrow Y$ ,  $f h = (\text{id} \# h) e$  implies  $f^\dagger h = e^\dagger$ ;



– *compositionality*: for every  $f : Y \rightarrow A \# Y$  and  $g : X \rightarrow Y \# X$  define

$$f^\dagger \bullet g = (X \xrightarrow{g} Y \# X \xrightarrow{f^\dagger \# \text{id}} A \# X)$$

and  $f \blacksquare g : Y + X \rightarrow A \# (Y + X)$  by

$$\begin{array}{ccc} Y + X & \xrightarrow{[u_Y^X, g]} & Y \# X \xrightarrow{f \# \text{id}} (A \# Y) \# X \\ & & \downarrow (\text{id} \# \text{inl}) \# \text{inr} \\ A \# (Y + X) & \xleftarrow{m_A^{Y+X}} & (A \# (Y + X)) \# (Y + X) \end{array}$$

Compositionality states that  $(f \blacksquare g)^\dagger \text{inr} = (f^\dagger \bullet g)^\dagger : X \rightarrow A$ .

A *morphism* from a complete Elgot  $\#$ -algebra  $(A, a, -^\dagger)$  to a complete Elgot  $\#$ -algebra  $(B, b, -^\dagger)$  is a  $\mathbf{C}$ -morphism  $f : A \rightarrow B$ , such that  $((f \# \text{id}) e)^\dagger = f e^\dagger$  for all  $e : X \rightarrow A \# X$ . This defines the category of complete  $\#$ -algebras  $\mathbf{CElg}_\#(\mathbf{C})$ .

*Remark 15.* Note that complete Elgot  $\#$ -algebras for the parametrized monad  $A \# X = A + \Sigma X$  (i.e. the parametrized monad of Example 9 (i) for  $\mathbb{T}$  the identity monad) are precisely the complete Elgot algebras for the functor  $\Sigma$  introduced and studied in [5].

**Example 16.** Let us come back to the three simple parameterized monads on  $\mathbf{Set}$  in Example 11 whose algebras are in each case simply binary algebras. In each of the three cases, morphisms  $X \rightarrow A \# X$  can be understood as a systems of mutual recursive equations of a certain type with variables from the set  $X$ , and the  $-^\dagger$  operation of a complete Elgot  $\#$ -algebra provides a solution of a given system of equations. However, the type of these systems of recursive equations is different for each of the three parameterized monads. For  $A \#_1 X = A + X \times X$ ,  $e : X \rightarrow A + X \times X$  can be understood as specifying for every  $x \in X$  precisely one equation of one of the two types below:

$$x \approx x' * x'' \quad \text{with } x', x'' \in X \quad \text{or} \quad x \approx a \quad \text{with } a \in A.$$

The solution  $e^\dagger$  then provides for every  $x \in X$  an element  $x^\dagger \in A$  that, when plugged into the above formal equations, turn them into identities in  $A$  where  $*$  is interpreted as the binary operation of  $A$ .

For  $A \#_2 X = A \times X^*$ , to give a morphism  $e : X \rightarrow A \times X^*$  is equivalent to give a system of recursive equations which specifies for each  $x \in X$  an equation

$$x \approx a * x' \quad \text{with } x' \in X, a \in A,$$

i.e. iteration is restricted to the second argument.

Finally, for  $A \#_3 X = A \times X^*$ , morphisms  $e : X \rightarrow BA$  simply specify for each  $x \in X$  a binary tree  $e(x)$ , and by the solution axiom,  $e^\dagger(x)$  is then the interpretation of this binary tree in  $A$ . Thus, iteration is trivial, in other words, every binary algebra is a complete Elgot algebra for  $\#_3$ .

**Example 17.** Continuous algebras are complete Elgot #-algebras. Consider any category  $\mathbf{C}$  that is enriched over  $\mathbf{Cppo}$  such that composition is left strict and a parameterized monad  $\#$  that is *locally continuous* in both arguments, i.e.  $\bigsqcup_i (f_i \# g_i) = (\bigsqcup_i f_i) \# (\bigsqcup_i g_i)$  holds for any  $f_i : A \rightarrow B$  and  $g_i : X \rightarrow Y$ . Then every #-algebra becomes a complete Elgot #-algebra when equipped with the operation  $-^\dagger$  assigning to every  $e : X \rightarrow A \# X$  its least solution. In more detail, let  $A$  be a #-algebra, to every  $e : X \rightarrow A \# X$  we assign  $e^\dagger : X \rightarrow A$  given by

$$e^\dagger = \bigsqcup_i e_i^\dagger,$$

where  $e_0^\dagger = \perp : X \rightarrow A$  and  $e_{i+1}^\dagger = a(\text{id} \# e_i^\dagger)e$ . That means that  $e^\dagger$  is the least fixed point of the function  $s \mapsto a(\text{id} \# s)e$  on  $\text{Hom}_{\mathbf{C}}(X, A)$ . The verification that this satisfies the axioms of a complete Elgot #-algebra can be found in the appendix.

Note that we did not require a morphism of complete Elgot #-algebras to be a morphism of #-algebras. Somewhat surprisingly, this follows automatically.

**Proposition 18.** *Let  $f : A \rightarrow B$  be a complete Elgot #-algebra morphism from  $(A, a, -^\dagger)$  to  $(B, b, -^\dagger)$ . Then  $f$  is a morphism of #-algebras.*

*Proof (Sketch).* The idea is to represent  $a$  as a loop terminating after the first iteration and then deduce preservation of  $a$  by  $f$  from preservation of iteration by  $f$  guaranteed by definition. More concretely, we take

$$e = (\text{id} \# \text{inr})[\text{id}, u_A^A] : (A \# A) + A \rightarrow A \# ((A \# A) + A)$$

and show that  $e^\dagger = [a, \text{id}]$ . The remaining proof amounts to deriving  $b(f \# f) = f a$  from  $f e^\dagger = ((f \# \text{id})e)^\dagger$ .  $\square$

It was shown by Uustalu [26] that parametrized monads give rise to monads at least in two different ways:

**Proposition 19.** *Suppose,  $\#$  is a parametrized monad on  $\mathbf{C}$  such that the least fixpoint  $\mu\gamma. X \# \gamma$  (the greatest fixpoint  $\nu\gamma. X \# \gamma$ ) exists for every  $X \in |\mathbf{C}|$ . Then  $\mu\gamma. - \# \gamma$  ( $\nu\gamma. - \# \gamma$ ) is the underlying functor of a monad.*

It is known that the initial algebra  $\mu\gamma. X \# \gamma$  yields the free #-algebra on  $X$ ; in fact, existence of this free #-algebra is equivalent to the existence of that initial algebra (see [7, Theorem 2.18]). Here we are interested in the final coalgebras  $\nu\gamma. X \# \gamma$ . These yield the free complete #-algebras, and moreover, existence of these free algebras is equivalent to the existence of that final coalgebras.

**Theorem 20.** *1. Suppose that  $\text{out}_X : F_\# X \rightarrow X \# F_\# X$  is a final  $(X \# -)$ -coalgebra. Then the following morphisms*

$$F_\# X \# F_\# X \xrightarrow{\text{out}_X \# \text{id}} (X \# F_\# X) \# F_\# X \xrightarrow{m_X^{F_\# X}} X \# F_\# X \xrightarrow{\text{out}_X^{-1}} F_\# X$$

and

$$X \xrightarrow{u_X^{F_\# X}} X \# F_\# X \xrightarrow{\text{out}_X^{-1}} F_\# X$$

form the algebra structure and universal morphism of a free complete Elgot algebra for  $\#$  on  $X$ .

2. Suppose that  $\varphi_X : FX \# FX \rightarrow FX$  and  $\eta_X : X \rightarrow FX$  form a free complete Elgot  $\#$ -algebra on  $X$ . Then

$$X \# FX \xrightarrow{\eta_X \# \text{id}} FX \# FX \xrightarrow{\varphi_X} FX$$

is an isomorphism, and its inverse is the structure of a final  $(X \# -)$ -coalgebra.

*Remark 21.* Note that in Clause (i) above the iteration operator on  $F_\# Y$  is obtained as follows. Given  $e : X \rightarrow F_\# Y \# X$  one forms the following coalgebra  $c : F_\# Y + X \rightarrow Y \# (F_\# Y + X)$  for  $Y \# -$ :

$$\begin{array}{ccc} F_\# Y + X & \xrightarrow{[u_{F_\# Y}^X, e]} & F_\# Y \# X \xrightarrow{\text{out} \# \text{id}} (Y \# F_\# Y) \# X \\ & & \downarrow (\text{id} \# \text{inl}) \# \text{inr} \\ Y \# (F_\# Y + X) & \xleftarrow{m_Y^{F_\# Y + X}} & (Y \# (F_\# Y + X)) \# (F_\# Y + X) \end{array}$$

Then one puts  $e^\dagger = (\text{coit } c) \text{ inr}$ .

The proof of Theorem 20 is a non-trivial generalization of the proof of [5, Theorem 5.4] from complete Elgot algebras for endofunctors to those for parameterized monads; we will establish Clause (i) as a consequence of Theorem 27 (see Corollary 28) while we outline the proof of Clause (ii) in the appendix.

Before we continue, let us note that, surprisingly, in a free complete Elgot algebra the iteration always assigns a unique solution to any  $e : X \rightarrow FX \# X$ .

**Proposition 22.** *Suppose that  $\varphi_Y : FY \# FY \rightarrow FY$  and  $\eta_Y : Y \rightarrow FY$  form a free complete Elgot  $\#$ -algebra on  $Y$ . Then for every  $e : X \rightarrow FY \# X$ ,  $e^\dagger : X \rightarrow FY$  is a unique solution, i.e. a unique morphism satisfying the solution axiom with  $e$ .*

From now on we assume that the final coalgebras  $\nu\gamma. X \# \gamma$  exist and denote them  $F_\# X$  (standardly omitting the structure morphisms  $\text{out}_X : F_\# X \rightarrow X \# F_\# X$ ). Recall that  $\text{coit } f : X \rightarrow F_\# Y$  is the morphism uniquely induced by a coalgebra  $(X, f : X \rightarrow Y \# X)$ . Following [26], in order to introduce and reason about the monad structure of  $F_\#$ , we use a more flexible *primitive corecursion principle*, derived from the standard *coiteration principle* embodied in  $\text{coit}$ .

**Proposition 23** ([26]). *For any functor  $F$  with a final coalgebra  $\nu F$ , and any  $f : X \rightarrow F(\nu F + X)$ , there is a unique morphism  $h$  satisfying  $\text{out } h = F[\text{id}, h] f$ .*

The morphism  $h$  in Proposition 23 is said to be defined by primitive corecursion. We use primitive corecursion to slightly generalize the  $\text{coit}$  construct in the special case of  $F_\#$ :

**Lemma 24.** *For any  $e : X \rightarrow B \# X$  and  $f : B \rightarrow A \# F_{\#}A$ , there is a unique morphism  $h$  satisfying*

$$\begin{array}{ccc} X & \xrightarrow{e} & B \# X \\ h \downarrow & & \downarrow m_A^{F_{\#}A} (f \# h) \\ F_{\#}A & \xrightarrow{\text{out}} & A \# F_{\#}A. \end{array} \quad (1)$$

For any  $e : X \rightarrow B \# X$  and  $f : B \rightarrow A \# F_{\#}A$  we denote by

$$\text{coit}(e, f) : X \longrightarrow F_{\#}A$$

the unique  $h$  making diagram (1) commute. Using (1), the monad structure on  $F_{\#}$  can be given as follows:

$$\begin{aligned} \eta_X^\nu &= \text{out}^{-1} u_X^{F_{\#}X} = \text{coit } u_X^X \\ f^* &= \text{coit}((f \# \text{id}) \text{ out}, \text{out}) \quad \text{where } f : X \rightarrow F_{\#}Y \end{aligned}$$

This also defines  $\mu^\nu = \text{id}^* = \text{coit}(\text{out}, \text{out})$ . Note that, by Lemma 24,  $f^*$  is the unique morphism satisfying equation

$$\text{out } f^* = m_Y^{F_{\#}Y} (\text{out } f \# f^*) \text{ out}. \quad (2)$$

**Lemma 25.** *Let  $e : X \rightarrow B \# X$  and  $f : B \rightarrow A \# F_{\#}A$ . Then*

$$\text{coit}(e, f) = (\text{out}^{-1} f)^* (\text{coit } e).$$

As an easy corollary of Lemma 25 we obtain that  $\text{coit } e = \text{coit}(e, u_X^{F_{\#}X})$ ; indeed, we have

$$\text{coit}(e, u_X^{F_{\#}X}) = (\text{out}^{-1} u_X^{F_{\#}X})^* (\text{coit } e) = (\eta_X^\nu)^* (\text{coit } e) = \text{coit } e.$$

We state another useful property in the following lemma:

**Lemma 26.** *Let  $e : X \rightarrow B \# X$  and  $g : B \rightarrow C$ . Then*

$$F_{\#}g (\text{coit } e) = \text{coit}((g \# \text{id}) e).$$

The following theorem is our first main result. It establishes an equivalence of complete Elgot  $\#$ -algebras and  $F_{\#}$ -algebras.

**Theorem 27.** *For any parameterized monad  $\# : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ , the Eilenberg-Moore algebras of  $F_{\#} = \nu\gamma$ .  $-\# \gamma$  are precisely the complete Elgot  $\#$ -algebras. More precisely,  $\mathbf{C}^{F_{\#}}$  and  $\mathbf{CElgt}_{\#}(\mathbf{C})$  are isomorphic categories under the identical on morphisms isomorphism constructed as follows:*

- $\mathbf{C}^{F_{\#}} \rightarrow \mathbf{CElgt}_{\#}(\mathbf{C})$ : for a  $F_{\#}$ -algebra  $(A, \chi : F_{\#}A \rightarrow A)$  we define a  $\#$ -algebra  $(A, \chi \text{out}^{-1}(\text{id} \# \eta^\nu) : A \# A \rightarrow A, -^\dagger)$  with  $e^\dagger = \chi(\text{coit } e) : X \rightarrow A$  for any  $e : X \rightarrow A \# X$ .

- $\mathbf{CElg}_\#(\mathbf{C}) \rightarrow \mathbf{C}^{F_\#}$ : for a  $\#$ -algebra  $(A, a : A \# A \rightarrow A, -^\dagger)$  we define a  $F_\#$ -algebra  $(A, \text{out}^\dagger : F_\# A \rightarrow A)$ .

*Proof (Sketch).* For the direction from  $\mathbf{C}^{F_\#}$  to  $\mathbf{CElg}_\#(\mathbf{C})$  we have to verify the axioms of complete Elgot  $\#$ -algebras. The hardest case is that of the compositionality identity. We have on the one hand

$$\begin{aligned}
(f^\dagger \bullet g)^\dagger &= \chi \text{coit}(f^\dagger \bullet g) \\
&= \chi \text{coit}((\chi(\text{coit } f) \# \text{id}) g) \\
&= \chi \text{coit}((\chi \# \text{id})((\text{coit } f) \# \text{id}) g) \\
&= \chi(F_\# \chi) \text{coit}(((\text{coit } f) \# \text{id}) g) && // \text{ Lemma 26} \\
&= \chi \mu^\nu \text{coit}(((\text{coit } f) \# \text{id}) g) && // \chi \text{ is an } F_\# \text{-algebra} \\
&= \chi \text{coit}(((\text{coit } f) \# \text{id}) g, \text{out}), && // \text{ Lemma 25}
\end{aligned}$$

and on the other hand, by definition,

$$(f \blacksquare g)^\dagger \text{inr} = \chi \text{coit}(m_A^{Y+X}(((\text{id} \# \text{inl}) f) \# \text{inr})[u_Y^X, g]) \text{inr}.$$

Let us denote  $m_A^{Y+X}(((\text{id} \# \text{inl}) f) \# \text{inr})[u_Y^X, g]$  by  $h$ . By Lemma 24, it suffices to show the identity  $\text{out}(\text{coit } h) \text{inr} = m_A^{F_\# A}(\text{out} \# ((\text{coit } h) \text{inr}))(\text{coit } f \# \text{id}) g$ . The latter is easy to obtain from the auxiliary equation  $(\text{coit } h) \text{inl} = \text{coit } f$  whose proof is a routine.

For the direction from  $\mathbf{CElg}_\#(\mathbf{C})$  to  $\mathbf{C}^{F_\#}$ , we have to prove the two axioms of Eilenberg-Moore algebras. The harder one is  $\text{out}^\dagger F_\#(\text{out}^\dagger) = \text{out}^\dagger \mu^\nu$  and it is obtained from the instance of compositionality  $(\text{out} \blacksquare \text{out})^\dagger \text{inr} = (\text{out}^\dagger \bullet \text{out})^\dagger$  by establishing  $\text{out}^\dagger[\text{id}, \mu^\nu] = (\text{out} \blacksquare \text{out})^\dagger$  and  $\text{out}^\dagger F_\#(\text{out}^\dagger) = (\text{out}^\dagger \bullet \text{out})^\dagger$ . Further calculations ensure that the correspondence between  $\mathbf{CElg}_\#(\mathbf{C})$  and  $\mathbf{C}^{F_\#}$  is functional and moreover an isomorphism.  $\square$

**Corollary 28.** *Free complete Elgot  $\#$ -algebras exist for all objects  $A$  of  $\mathbf{C}$  if and only if the final coalgebras  $F_\# A$  exist. The functor assigning to an object  $A$  its free complete Elgot  $\#$ -algebra is  $FA = (F_\# A, \text{out}_A^{-1} m_A^{F_\# A}(\text{out}_A \# \text{id}_{F_\# A}), -^\dagger)$ , where  $-^\dagger$  is the iteration operation defined in Remark 21.*

## 5 Algebras of Complete Elgot Monads

We are now in a position to apply the results on complete Elgot  $\#$ -algebras developed in the previous section to explore the connection between complete Elgot monads and complete Elgot algebras. We briefly motivate our further technical contribution as follows.

Recall that given a monad  $\mathbb{T}$  and an endofunctor  $\Sigma$  over  $\mathbf{C}$ ,  $X \# Y = T(X + \Sigma Y)$  is a parametrized monad and therefore, by Proposition 19,  $\mathbb{T}_\Sigma$  given by  $(\star)$  is a monad. We reserve notation  $\mathbb{T}_\nu$  for the special case when  $\Sigma = \text{Id}$ :

$$T_\nu X = \nu \gamma. T(X + \gamma).$$

From a computational point of view,  $T_\nu X$  can be considered as a type of processes triggering a computational effect formalized by  $\mathbb{T}$  at each step and eventually outputting values from  $X$  in case of successful termination. The unary operation captured by  $\Sigma = \text{Id}$  intuitively means the action of *delaying*. This perspective was previously pursued in [15]. Now, if  $\mathbb{T}$  is a complete Elgot monad, or more generally, any monad equipped with an iteration operator, we can define a *collapsing morphism*  $\delta_X : T_\nu X \rightarrow TX$  as follows:

$$\delta_X = \left( T_\nu X \xrightarrow{\text{out}_X} T(X + T_\nu X) \right)^\dagger, \quad (3)$$

which intuitively flattens every possibly infinite sequence of computational steps of  $T_\nu X$  into a single step of  $TX$ . Let us illustrate this with the following toy example.

**Example 29.** Let  $TX = \mathcal{P}_{\omega_1}(A^* \times X)$  where  $\mathcal{P}_{\omega_1}$  is the countable powerset functor and  $A$  is some fixed alphabet of *actions* like in Example 1. We extend  $T$  to a monad  $\mathbb{T}$  by putting

$$\eta_X(x) = \{(\varepsilon, x)\} \quad \text{and} \quad f^*(s \subseteq A^* \times X) = \{(ww', y) \mid (w, x) \in s, (w', y) \in f(x)\},$$

where  $\varepsilon \in A^*$  is the empty word and  $f : X \rightarrow \mathcal{P}_{\omega_1}(A^* \times Y)$ . It is easy to see that  $\mathbb{T}$  is an  $\omega$ -continuous monad (see Example 3) and hence a complete Elgot monad with the iteration operator defined using least fixed points. An element of  $TX$  is intuitively a countably branching process, with results in  $X$ , at each step capable of executing a finite series of actions. Now the collapsing morphism (3) for every process  $p \in T_\nu\{\checkmark\}$  calculates the set  $\text{tr}(p) \subseteq A^*$  of all successful traces of  $p$ .

As we will see later (Theorem 32 (i)),  $(TX, T_\nu TX \xrightarrow{\delta_{TX}} TTX \xrightarrow{\mu_X} TX)$  is a  $\mathbb{T}_\nu$ -algebra and hence, by Theorem 27, a complete Elgot  $\#$ -algebra. We can now change the perspective and instead of  $TX$  consider an arbitrary complete Elgot  $\#$ -algebra. The question we consider next is: Is it possible to recover the laws of iteration for  $\mathbb{T}$  assuming that every  $\mathbb{T}$  is coherently equipped with the structure of a complete Elgot  $\#$ -algebra? It turns out that without any further assumptions on the category of complete Elgot  $\#$ -algebras almost all laws of complete Elgot monads become derivable. More precisely, we introduce the following class of monads.

**Definition 30.** A monad  $\mathbb{T}$  is called a *weak complete Elgot monad* if it is equipped with an iteration operator  $-^\dagger$  that satisfies *fixpoint*, *naturality*, and *uniformity* axioms and the following identity: for any  $g : X \dashv\dashv Y + X$ ,  $f : Y \dashv\dashv Z + Y$  we have

$$\left( Y + X \xrightarrow{[\text{inl}, g]} Y + X \xrightarrow{f + \text{id}} Z + Y + X \right)^\dagger \text{inr} = X \xrightarrow{g^\dagger} Y \xrightarrow{f^\dagger} Z. \quad (4)$$

(See Fig. 3 for the pictorial form.)

It is relatively easy to deduce (4) from the codiagonal identity, hence we obtain

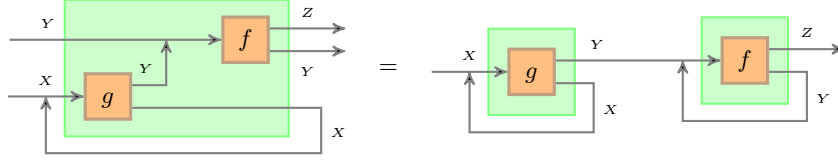


Fig. 3: The additional axiom for weak complete Elgot monads.

**Proposition 31.** *Any complete Elgot monad is a weak complete Elgot monad.*

We now can establish a tight connection between weak complete Elgot monads and complete Elgot  $\#$ -algebras.

**Theorem 32.** *Let  $\mathbb{T}$  be a monad on  $\mathbf{C}$  and let  $X \# Y = T(X + Y)$ .*

1. *If  $\mathbb{T} = (T, \eta, -^*, -^\dagger)$  is a weak complete Elgot monad then  $\mathbf{C}^\mathbb{T}$  is isomorphic to the full subcategory of  $\mathbf{CElg}_\#(\mathbf{C})$  formed by those complete Elgot  $\#$ -algebras  $(A, a : T(A + A) \rightarrow A, -^\dagger)$  which factor through  $T\nabla : T(A + A) \rightarrow TA$  and for which  $e^\dagger = a(T\text{inl})e^\dagger$  for every  $e : X \rightarrow T(A + X)$ .*
2. *Conversely, any functor  $J : \mathbf{C}^\mathbb{T} \rightarrow \mathbf{CElg}_\#(\mathbf{C})$  sending a  $\mathbb{T}$ -algebra  $a : TA \rightarrow A$  to  $a(T\nabla) : T(A + A) \rightarrow A$  and identical on morphisms induces a weak complete Elgot monad structure on  $\mathbb{T}$  as follows:*

$$\frac{e : X \rightarrow T(Y + X)}{e^\dagger = (T(\eta + \text{id})e)^\dagger : X \rightarrow TY} \quad (5)$$

where  $-^\dagger$  is the iteration operator on  $J(TY, \mu)$  (by Clause (i),  $J$  is then full and faithful).

If  $\mathbf{CElg}_\#(\mathbf{C})$  additionally satisfy a version of the codiagonal identity, the construction from Clause (ii) of Theorem 32 produces precisely complete Elgot monads.

**Theorem 33.** *Let  $\mathbb{T}$  be a monad on  $\mathbf{C}$ , let  $X \# Y = T(X + Y)$  and let  $J : \mathbf{C}^\mathbb{T} \rightarrow \mathbf{CElg}_\#(\mathbf{C})$  be a functor as in Clause (ii) of Theorem 32. Then  $\mathbb{T}$  is equipped with the structure of a weak complete Elgot monad given by (5), and moreover  $\mathbb{T}$  is a complete Elgot monad iff every  $(A, a, -^\dagger)$  in  $\mathbf{CElg}_\#(\mathbf{C})$  satisfies the equation*

$$(m_{A\#X}^X e)^\dagger = (e^\dagger)^\dagger \quad (6)$$

for every  $e : X \rightarrow (A \# X) \# X$  (this uses the fact that  $A \# X = T(A + X)$  is a free  $\mathbb{T}$ -algebra and hence a complete Elgot  $\#$ -algebra).

## 6 Conclusions and Further Work

We introduced the notion of complete Elgot algebra for a parametrized monad, based on the previous work [4, 26]. We showed that the category of complete

Elgot algebras for a parametrized monad  $\#$  is isomorphic to the category of Eilenberg-Moore algebras for the monad  $\nu\gamma. - \# \gamma$  whenever the latter exists. As the category of complete Elgot  $\#$ -algebras is given axiomatically, this can be considered as a form of soundness and completeness result, specifically, it indicates that algebras for  $\nu\gamma. - \# \gamma$  are subject to a lightweight theory of (uniform) iteration.

We explored the connection between complete Elgot  $\#$ -algebras for  $X \# Y = T(X + Y)$  and Eilenberg-Moore algebras of complete Elgot monads, i.e. monads from [14] supporting a uniform iteration operator satisfying standard axioms of iteration. Specifically, we showed that monads  $\mathbb{T}$  whose algebras are coherently equipped with the structure of a complete Elgot  $\#$ -algebra are precisely complete Elgot monads with the codiagonal axiom replaced by its weakened form (Theorem 32). Moreover, if the category of complete Elgot  $\#$ -algebras satisfies a variant of the codiagonal law, such monads  $\mathbb{T}$  are complete Elgot monads (Theorem 33).

As further work we plan to improve Theorem 33 to obtain an intrinsic characterization of complete Elgot monads in the style of Theorem 32 (i.e. without assuming extra properties of the complete Elgot algebras). We believe that the results we obtained are potentially useful for facilitating constructions over complete Elgot monads, in particular we are seeking for a conceptual simplification for the sophisticated proofs underlying the main result of [14] stating that  $(\star)$  is a complete Elgot monad whenever  $\mathbb{T}$  is. Also we are interested in applications of the obtained results to semantics of abstract side-effecting processes in the style of [15].

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## A Appendix: Omitted proofs

### Proof of Proposition 6 (Bekić identity)

Let us show that complete Elgot monads validate the Bekić identity. Let

$$u = T((\text{id} + \text{inl}) + \text{inr}) [f, g] : Y + X \rightarrow T((Z + (Y + X)) + (Y + X)).$$

By *codiagonal*,

$$(T[\text{id}, \text{inr}] u)^\dagger = (u^\dagger)^\dagger. \quad (7)$$

Now the left-hand side of (7) simplifies to

$$\begin{aligned} & (T[\text{id}, \text{inr}] T((\text{id} + \text{inl}) + \text{inr}) [f, g])^\dagger \\ &= (T[\text{id} + \text{inl}, \text{inr inr}] [f, g])^\dagger \\ &= (T\alpha [f, g])^\dagger, \end{aligned}$$

i.e. to the left-hand side of the Bekić identity. Now observe that, by *uniformity* and *naturality*,

$$u^\dagger \text{ inr} = (T(\text{id} + \text{inl}) + \text{id}) g^\dagger = T(\text{id} + \text{inl}) g^\dagger. \quad (8)$$

Therefore, the right-hand side of (7) can be rewritten in the form

$$\begin{aligned} (u^\dagger)^\dagger &= ([\eta, u^\dagger]^\star u)^\dagger && // \text{fixpoint} \\ &= ([\eta (\text{id} + \text{inl}), u^\dagger \text{ inr}] [f, g])^\dagger \\ &= ([T(\text{id} + \text{inl}) \eta, T(\text{id} + \text{inl}) g^\dagger]^\star [f, g])^\dagger && // (8) \\ &= (T(\text{id} + \text{inl}) [\eta, g^\dagger]^\star [f, g])^\dagger \\ &= (T(\text{id} + \text{inl}) [[\eta, g^\dagger]^\star f, g^\dagger])^\dagger && // \text{fixpoint} \\ &= ([\eta \text{ inl}, \eta \text{ inr inl}]^\star [[\eta, g^\dagger]^\star f, g^\dagger])^\dagger \\ &= [\eta, ([\eta \text{ inl}, [[\eta, g^\dagger]^\star f, g^\dagger]]^\star \eta \text{ inr inl})^\dagger]^\star \\ &\quad [[\eta, g^\dagger]^\star f, g^\dagger] && // \text{dinaturality} \\ &= [\eta, ([\eta, g^\dagger]^\star f)^\dagger]^\star [[\eta, g^\dagger]^\star f, g^\dagger] \\ &= [([\eta, g^\dagger]^\star f)^\dagger, [\eta, ([\eta, g^\dagger]^\star f)^\dagger]^\star g^\dagger] && // \text{fixpoint} \\ &= [h^\dagger, [\eta, h^\dagger]^\star g^\dagger] \\ &= [\eta, h^\dagger]^\star [\eta \text{ inr}, g^\dagger], \end{aligned}$$

i.e. equals the right-hand side of the Bekić identity.

For the opposite direction, we need to show that the Bekić identity implies *dinaturality* and *codiagonal*. For the latter, let  $k : X \rightarrow T((Y + X) + X)$ . By the Bekić identity,

$$(T\alpha [k, k])^\dagger = [\eta, ([\eta, k^\dagger]^\star k)^\dagger]^\star [\eta \text{ inr}, k^\dagger]$$

$$= [\eta, (k^\dagger)^\dagger]^\star [\eta \text{ inr}, k^\dagger].$$

Thus,  $(T\alpha[k, k])^\dagger \text{ inl} = (T\alpha[k, k])^\dagger \text{ inr} = (k^\dagger)^\dagger$ . On the other hand, by *uniformity*,

$$(T\alpha[k, k])^\dagger = (T[\text{id}, \text{inr}] k)^\dagger [\text{id}, \text{id}]$$

and therefore

$$(k^\dagger)^\dagger = (T\alpha[k, k])^\dagger \text{ inr} = (T[\text{id}, \text{inr}] k)^\dagger$$

as required. To prove *dinaturality*, we define the term

$$w = ([T(\text{id} + \text{inr}) h, T(\text{id} + \text{inl}) g])^\dagger$$

for  $g : X \rightarrow T(Y + Z)$ ,  $h : Z \rightarrow T(Y + X)$ . By uniformity,

$$w [\text{inr}, \text{inl}] = ([T(\text{id} + \text{inr}) g, T(\text{id} + \text{inl}) h])^\dagger.$$

The Bekić identity then gives us

$$\begin{aligned} w &= ([T(\text{id} + \text{inr}) h, T(\text{id} + \text{inl}) g])^\dagger \\ &= (T\alpha[T(\text{inl} + \text{id}) h, T \text{ inl } g])^\dagger \\ &= [\eta, ([\eta, (T \text{ inl } g)^\dagger]^\star T(\text{inl} + \text{id}) h)^\dagger]^\star [\eta \text{ inr}, (T \text{ inl } g)^\dagger] \\ &= [\eta, ([\eta \text{ inl}, g]^\star h)^\dagger]^\star [\eta \text{ inr}, g] \end{aligned}$$

as well as

$$w [\text{inr}, \text{inl}] = [\eta, ([\eta \text{ inl}, h]^\star g)^\dagger]^\star [\eta \text{ inr}, h].$$

Therefore,

$$([\eta \text{ inl}, h]^\star g)^\dagger = w [\text{inr}, \text{inl}] \text{ inl} = w \text{ inr} = [\eta, ([\eta \text{ inl}, g]^\star h)^\dagger]^\star g. \quad \square$$

### Proof of Proposition 12

We define constructions to convert from  $\#$ -algebras to  $\mathbb{T}$ - $\Sigma$ -bialgebras and vice versa.

1. Given a  $\#$ -algebra  $\alpha : T(A + \Sigma A) \rightarrow A$ , let

$$\begin{aligned} a &= TA \xrightarrow{T \text{ inl}} T(A + \Sigma A) \xrightarrow{\alpha} A \\ f &= \Sigma A \xrightarrow{\eta \text{ inr}} T(A + \Sigma A) \xrightarrow{\alpha} A. \end{aligned}$$

We immediately check that  $a$  satisfies the properties of a  $\mathbb{T}$ -algebra:

$$\begin{aligned}
a\eta &= \alpha(T \text{ inl}) \eta & a\mu &= \alpha(T \text{ inl}) \mu \\
&= \alpha \eta \text{ inl} & &= \alpha \mu (TT \text{ inl}) \\
&= \alpha u & &= \alpha \mu T[\text{id}, \eta \text{ inr}] (T \text{ inl}) (TT \text{ inl}) \\
&= \text{id} & &= \alpha m (T \text{ inl}) (TT \text{ inl}) \\
& & &= \alpha T(\alpha + \text{id}) (T \text{ inl}) (TT \text{ inl}) \\
& & &= \alpha (T \text{ inl}) (T\alpha) (TT \text{ inl}) \\
& & &= a(Ta)
\end{aligned}$$

2. Conversely, given a bialgebra  $TA \xrightarrow{a} A \xleftarrow{f} \Sigma A$ , form

$$\alpha = T(A + \Sigma A) \xrightarrow{T[\text{id}, f]} TA \xrightarrow{a} A.$$

The constructed  $\alpha$  is a  $\#$ -algebra:

$$\begin{aligned}
\alpha u &= aT[\text{id}, f] \eta \text{ inl} & \alpha m &= aT[\text{id}, f] \mu T[\text{id}, \eta \text{ inr}] \\
&= a\eta & &= a\mu TT[\text{id}, f] T[\text{id}, \eta \text{ inr}] \\
&= \text{id} & &= aTaTT[\text{id}, f] T[\text{id}, \eta \text{ inr}] \\
& & &= aTaT[T[\text{id}, f], \eta f] \\
& & &= aT[aT[\text{id}, f], f] \\
& & &= aT[\text{id}, f] T[\text{inl } aT[\text{id}, f], \text{inr}] \\
& & &= \alpha T(\alpha + \text{id})
\end{aligned}$$

Next, we show that the passages (i) and (ii) are mutually inverse.

- From bialgebras to  $\#$ -algebras and back: Given the bialgebra  $TA \xrightarrow{a} A \xleftarrow{f} \Sigma A$ , constructing  $\alpha = aT[\text{id}, f]$  as in (ii), one obtains back  $a$  and  $f$  using (i):

$$\begin{aligned}
\alpha(T \text{ inl}) &= aT[\text{id}, f] (T \text{ inl}) = a \\
\alpha \eta \text{ inr} &= aT[\text{id}, f] \eta \text{ inr} = a\eta f = f
\end{aligned}$$

- From  $\#$ -algebras to bialgebras and back: Given  $\alpha : T(A + \Sigma A) \rightarrow A$ , we construct  $a = \alpha(T \text{ inl})$  and  $f = \alpha \eta \text{ inr}$  and obtain:

$$\begin{aligned}
aT[\text{id}, f] &= \alpha(T \text{ inl}) T[\text{id}, \alpha \eta \text{ inr}] \\
&= \alpha(T \text{ inl}) T[\alpha u, \alpha \eta \text{ inr}] \\
&= \alpha(T \text{ inl}) T[\alpha \eta \text{ inl}, \alpha \eta \text{ inr}] \\
&= \alpha(T \text{ inl}) (T\alpha) (T\eta) \\
&= \alpha T(\alpha + \text{id}) (T \text{ inl}) (T\eta) \\
&= \alpha m (T \text{ inl}) (T\eta)
\end{aligned}$$

$$\begin{aligned}
&= \alpha \mu T[\text{id}, \eta \text{ inr}] (T \text{ inl}) (T \eta) \\
&= \alpha \mu (T \eta) \\
&= \alpha
\end{aligned}$$

□

### Details for Example 17

We verify the three axioms of complete Elgot  $\#$ -algebras for  $A$  equipped with the least solution.

- solution: this clearly holds because  $e^\dagger$  is a fixed point of the map  $s \mapsto a(\text{id} \# s) e$ .
- uniformity: let  $e : X \rightarrow A \# X$ ,  $f : Y \rightarrow A \# Y$  and  $h : X \rightarrow Y$  such that  $fh = (\text{id} \# h)e$  holds. In order to show that  $f^\dagger h = e^\dagger$  we show by induction that for every  $i$  we have

$$f_i^\dagger h = e_i^\dagger.$$

The base case is clear: since composition is left strict we have  $\perp h = \perp$ . For the induction step we compute:

$$\begin{aligned}
f_{i+1}^\dagger h &= a(\text{id} \# f_i^\dagger) f h \\
&= a(\text{id} \# f_i^\dagger) (\text{id} \# h) e \\
&= a(\text{id} \# f_i^\dagger h) e \\
&= a(\text{id} \# e_i^\dagger) e \\
&= e_{i+1}^\dagger.
\end{aligned}$$

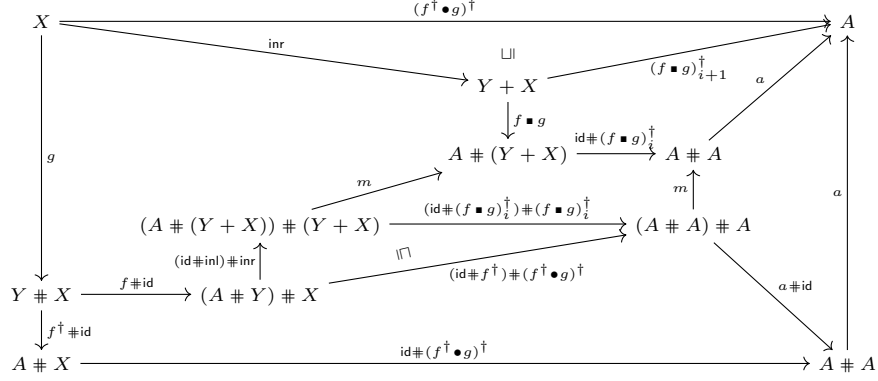
- compositionality: let  $f : Y \rightarrow A \# Y$  and  $g : X \rightarrow Y \# X$ . The desired equation  $(f \blacksquare g)^\dagger \text{ inr} = (f^\dagger \bullet g)^\dagger$  is established by proving by induction the following two inequalities for every  $i$ :

$$(f \blacksquare g)_i^\dagger \text{ inr} \sqsubseteq (f^\dagger \bullet g)^\dagger \tag{9}$$

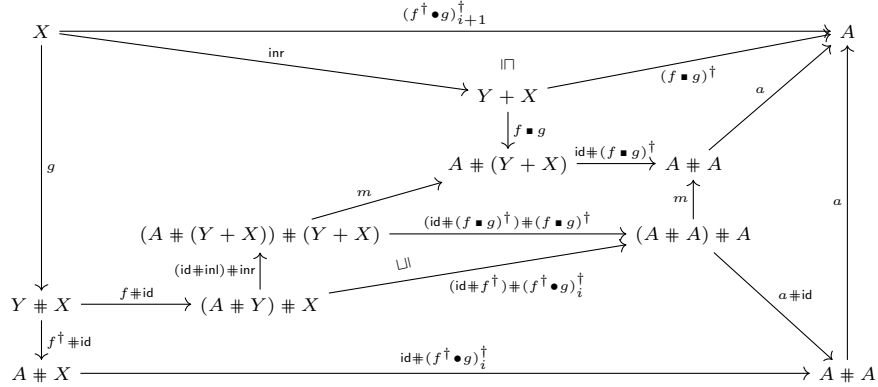
$$(f^\dagger \bullet g)_i^\dagger \sqsubseteq (f \blacksquare g)^\dagger \text{ inr} \tag{10}$$

For (9), the base case is clear by left strictness. For the induction step first note that  $(f \blacksquare g) \text{ inl} = (\text{id} \# \text{inl})f$ , thus  $(f \blacksquare g)^\dagger \text{ inl} = f^\dagger$  by uniformity. Now we

consider the following diagram



We are to prove the inequality in the upper triangle. We start with the inequality in the middle triangle; it holds by the induction hypothesis and since  $(f \blacksquare g)_i^\dagger \text{inl} \sqsubseteq (f \blacksquare g)^\dagger \text{inl} = f^\dagger$ . The other inner parts clearly commute; for the lower square consider the left- and right-hand components of  $\#$  separately: the right-hand one commutes trivially, and for the left-hand one use the solution axiom for  $f$ . Since the outside of the diagram also commutes by the solution axiom, we obtain the desired inequality in the upper triangle. For (10), the base case is clear once again. For the induction step we consider the diagram below:



We are to prove the inequality in the upper triangle. We start with the inequality in the middle triangle; it holds by the induction hypothesis and since  $(f \blacksquare g)^\dagger \text{inl} = f^\dagger$ . The other inner parts commute as in the previous diagram. Finally, the outside commutes by the definition of  $(f^\dagger \bullet g)^\dagger_{i+1}$ . Thus, we obtain the inequality in the upper triangle as desired.  $\square$

### Proof of Proposition 18

Let  $e = (\text{id} \# \text{inr}) [\text{id}, u_A^A] : (A \# A) + A \rightarrow A \# ((A \# A) + A)$ . We show that  $e^\dagger = [a, \text{id}]$ , and therefore  $e^\dagger \text{inl} = a$ . To that end we successively calculate  $e^\dagger \text{inr}$

and  $e^\dagger \text{ inl}$ :

$$\begin{aligned}
e^\dagger \text{ inr} &= a(\text{id} \# e^\dagger) e \text{ inr} && // \text{ solution} \\
&= a(\text{id} \# e^\dagger) (\text{id} \# \text{inr}) u_A^A \\
&= a(\text{id} \# e^\dagger) u_A^{(A \# A) + A} \text{id} \\
&= a u_A^A \text{id} \\
&= \text{id}. \\
e^\dagger \text{ inl} &= a(\text{id} \# e^\dagger) e \text{ inl} && // \text{ solution} \\
&= a(\text{id} \# e^\dagger) (\text{id} \# \text{inr}) \\
&= a(\text{id} \# (e^\dagger \text{ inr})) \\
&= a(\text{id} \# \text{id}) \\
&= a.
\end{aligned}$$

To finish the proof, let us show that the following diagram:

$$\begin{array}{ccccc}
A \# A & \xrightarrow{\text{inl}} & (A \# A + A) & \xrightarrow{e^\dagger = [a, \text{id}]} & A \\
\downarrow f \# f & & & \searrow ((f \# \text{id}) e)^\dagger & \downarrow f \\
B \# B & \xrightarrow{b} & & & B
\end{array}$$

commutes. The triangle commutes since  $f$  is a morphism of  $\#$ -algebras, and we are left to show commutativity of the inner quadrangle. Observe that

$$\begin{aligned}
((f \# \text{id}) e)^\dagger \text{ inr} &= b(\text{id} \# ((f \# \text{id}) e)^\dagger) (f \# \text{id}) e \text{ inr} && // \text{ solution} \\
&= b(\text{id} \# ((f \# \text{id}) e)^\dagger) (f \# \text{inr}) u_A^A \\
&= b(f \# ((f \# \text{id}) e)^\dagger) u_A^{(A \# A) + A} \\
&= b u_B^B f \\
&= f
\end{aligned}$$

from which we conclude the desired identity:

$$\begin{aligned}
((f \# \text{id}) e)^\dagger \text{ inl} &= b(\text{id} \# ((f \# \text{id}) e)^\dagger) (f \# \text{id}) e \text{ inl} && // \text{ solution} \\
&= b(\text{id} \# ((f \# \text{id}) e)^\dagger) (f \# \text{inr}) \\
&= b(f \# (((f \# \text{id}) e)^\dagger \text{ inr})) \\
&= b(f \# f). \quad \square
\end{aligned}$$

### Proofsketch for Theorem 20

Before we outline the proof of the desired result, we explain an auxiliary construction that produces from a given complete Elgot algebra  $A$  and morphism  $f : Y \rightarrow A$  a new complete Elgot algebra on  $Y \# A$ .

**Construction 34** Let  $(A, \alpha, -^\dagger)$  a complete Elgot  $\#$ -algebra and let  $m : Y \rightarrow A$  be a morphism. Then form the following morphism

$$\alpha^f = \left( (Y \# A) \# (Y \# A) \xrightarrow{\text{id} \# (f \# \text{id})} (Y \# A) \# (A \# A) \xrightarrow{\text{id} \# \alpha} (Y \# A) \# A \xrightarrow{m_Y^A} Y \# A \right)$$

and define the dagger operation  $-^\dagger$  as follows: given  $e : X \rightarrow (Y \# A) \# X$  one forms

$$\bar{e} = (X \xrightarrow{e} (Y \# A) \# X \xrightarrow{(f \# \text{id}) \# \text{id}} (A \# A) \# X \xrightarrow{\alpha \# \text{id}} A \# X),$$

i.e.  $\bar{e} = (\alpha(f \# \text{id})) \bullet e$ , and then one puts

$$e^\dagger = \left( X \xrightarrow{e} (Y \# A) \# X \xrightarrow{\text{id} \# \bar{e}^\dagger} (Y \# A) \# A \xrightarrow{m_Y^A} Y \# A \right).$$

**Lemma 35.** The triple  $(Y \# A, \alpha^f, -^\dagger)$  is a complete Elgot  $\#$ -algebra such that

$$Y \# A \xrightarrow{f \# \text{id}} A \# A \xrightarrow{\alpha} A$$

is a morphism of complete Elgot algebras.

The proof is a somewhat involved computation adapted from the proof of [5, Lemma 5.6]. The previous lemma and the following proposition provide a Lambek-type lemma for complete Elgot  $\#$ -algebras.

**Proposition 36.** If  $(FY, \varphi_Y, -^\dagger)$  is a free complete Elgot  $\#$ -algebra on  $Y$  with universal morphism  $\eta_Y : Y \rightarrow FY$ , then

$$Y \# FY \xrightarrow{\eta_Y \# \text{id}} FY \# FY \xrightarrow{\varphi_Y} FY$$

is an isomorphism.

*Proof.* By Lemma 35,  $Y \# FY$  with algebra structure and  $-^\dagger$  formed as in Construction 34 (for  $A = FY$ ,  $\alpha = \varphi_Y$  and  $f = \eta_Y$ ) is a complete Elgot  $\#$ -algebra. By the freeness of  $FY$ , we obtain a unique morphism of complete Elgot  $\#$ -algebras  $t : FY \rightarrow Y \# FY$  such that  $t\eta_Y = u_Y^{FY} : Y \rightarrow Y \# FY$ . Let us denote  $t' = \varphi_Y(\eta_Y \# \text{id})$ . Then it is our task to prove that  $t$  and  $t'$  are mutually inverse.

Indeed, we have  $t't = \text{id}_{FY}$  since both  $t$  and  $t'$  are morphisms of complete Elgot  $\#$ -algebras and since

$$t't\eta_Y = t'u_Y^{FY} = \varphi_Y(\eta_Y \# \text{id})u_Y^{FY} = \varphi_Y u_{FY}^{FY} \eta_Y = \eta_Y$$

using naturality of  $u$  and the unit law of the  $\#$ -algebra structure  $\varphi_Y$ . The freeness of  $FY$  now yields the desired equation.



In order to prove  $t t' = \text{id}_{Y \# FY}$  notice first that  $t$  is a morphism of  $\#$ -algebras by Proposition 18. Now consider the diagram below:

$$\begin{array}{ccccc}
 Y \# FY & \xrightarrow{\text{id} \# t} & Y \# (Y \# FY) & & \\
 \downarrow \eta_Y \# \text{id} & & \swarrow u_Y^{FY} \# \text{id} & & \downarrow \\
 FY \# FY & \xrightarrow{t \# t} & (Y \# FY) \# (Y \# FY) & \xrightarrow{\text{id} \# (\eta_Y \# \text{id})} & Y \# (FY \# FY) \\
 \downarrow \varphi_Y & & \downarrow \text{id} \# (\eta_Y \# \text{id}) & & \downarrow \text{id} \# \varphi_Y \\
 & & (Y \# FY) \# (FY \# FY) & \xleftarrow{u_{FY}^{FY} \# \text{id}} & Y \# (FY \# FY) \\
 & & \downarrow \text{id} \# \varphi_Y & & \downarrow \text{id} \# \varphi_Y \\
 & & (Y \# FY) \# FY & \xleftarrow{u_Y^{FY} \# \text{id}} & Y \# FY \\
 & & \downarrow m_Y^{FY} & & \uparrow \text{id} \\
 FY & \xrightarrow{t} & Y \# FY & \xleftarrow{\text{id}} & Y \# FY
 \end{array}$$

$t'$  (left vertical arrow from  $Y \# FY$  to  $FY$ )  
 $t$  (bottom horizontal arrow from  $FY$  to  $Y \# FY$ )  
 $\text{id} \# t'$  (right vertical arrow from  $Y \# (Y \# FY)$  to  $Y \# FY$ )

All its inner parts commute: the left- and right-hand parts commute by the definition of  $t'$ , the big lower left-hand part commutes since  $t$  is a  $\#$ -algebra morphism, the upper part commutes using that  $t \eta_Y = u_Y^{FY}$ , the lower right-hand triangle commutes by the monad laws for  $- \# FY$  and the remaining three parts are obvious. Thus, the outside commutes, and since  $t' t = \text{id}$  we conclude that  $t t' = \text{id}$ , which completes the proof.

Let us henceforth denote for a given free complete Elgot  $\#$ -algebra on  $Y$

$$t = (\varphi_Y(\eta_Y \# \text{id}))^{-1} : FY \rightarrow Y \# FY.$$

**Corollary 37.** *The following square commutes:*

$$\begin{array}{ccc}
 FY \# FY & \xrightarrow{\varphi_Y} & FY \\
 \downarrow t \# \text{id} & & \downarrow t \\
 (Y \# FY) \# FY & \xrightarrow{m_Y^{FY}} & Y \# FY
 \end{array}$$

*Proof.* Use that  $t$  is a morphism of  $\#$ -algebras, i.e. consider the big lower left-hand part of the diagram in the proof of Proposition 36. Then use that  $\varphi_Y(\eta_Y \# \text{id})t = \text{id}$  implies that

$$(\text{id} \# \varphi_Y)(\text{id} \# (\eta_Y \# \text{id}))(t \# t) = t \# \text{id}$$

to obtain the commutativity of the desired square.

We are now ready to prove item 2. of Theorem 20. So suppose that  $(FY, \varphi_Y, -^\dagger)$  is a free complete Elgot  $\#$ -algebra on  $Y$  with universal morphism  $\eta_Y : Y \rightarrow FY$ .

By Lemma 35, we know that  $\varphi_Y(\eta_Y \# \text{id}) : Y \# FY \rightarrow FY$  is an isomorphism with inverse  $t : FY \rightarrow Y \# FY$ . One then proves that  $(FY, t)$  is a final coalgebra for  $Y \# -$ .

Indeed, given any coalgebra  $c : X \rightarrow Y \# X$  one forms the equation morphism

$$e = (X \xrightarrow{c} Y \# X \xrightarrow{\eta_Y \# \text{id}} FY \# X).$$

Then it is easy to see that  $e^\dagger : X \rightarrow FY$  is a coalgebra homomorphism from  $(X, c)$  to  $(FY, t)$ ; in fact, consider the diagram below:

$$\begin{array}{ccccc} X & \xrightarrow{e^\dagger} & FY & & \\ \downarrow c & \searrow e & \uparrow \varphi_Y & & \\ & FY \# X & \xrightarrow{FY \# e^\dagger} & FY \# FY & \\ & \uparrow \eta_Y \# \text{id} & & \uparrow \eta_Y \# \text{id} & \\ Y \# X & \xrightarrow{\text{id} \# e^\dagger} & Y \# FY & \xleftarrow{t} & \end{array}$$

All its inner parts commute: the upper part commutes since  $e^\dagger$  is a solution of  $e$ , the left-hand triangle commutes by the definition of  $e$ , the lower part commutes trivially, and for the right-hand part use that  $t$  is the inverse of  $\varphi_Y(\eta_Y \# \text{id})$ . It remains to prove that uniqueness of a coalgebra homomorphism from  $(X, c)$  to  $(FY, t)$ . This proof can be performed analogously to the proof of part (2)  $\Rightarrow$  (1) of [5, Theorem 5.4].  $\square$

### Proof of Proposition 22

Recall first from Theorem 20 that  $FY$  is (equivalently) a final  $(Y \# -)$ -coalgebra with the structure  $t : FY \rightarrow Y \# FY$  obtained as an inverse of

$$Y \# FY \xrightarrow{\eta_Y \# FY} FY \# FY \xrightarrow{\varphi_Y} FY.$$

Let  $e : X \rightarrow FY \# X$  and consider the following  $(Y \# -)$ -coalgebra

$$\begin{aligned} \bar{e} &= (FY \# X \xrightarrow{t \# e} (Y \# FY) \# (FY \# X)) \\ &\quad \downarrow (Y \# u_{FY}^X) \# (FY \# Y) \\ &= (Y \# (FY \# X)) \# (FY \# X) \xrightarrow{m_Y^{FY \# X}} Y \# (FY \# X). \end{aligned}$$

Now let  $d : X \rightarrow FY$  be any solution of  $e$ , i.e. we have  $d = \varphi_Y(FY \# d)e$ . We will prove below that  $\varphi_Y(FY \# d) : FY \# X \rightarrow FY$  is a coalgebra homomorphism from  $\bar{e}$  to  $t$ . Since  $\bar{e}$  does not depend on the solution  $d$  we then conclude that

$$e^\dagger = \varphi_Y(FY \# e^\dagger)e = \varphi_Y(FY \# d)e = d$$

using finality of  $FY$  in the middle step.

To finish the proof consider the following diagram:

$$\begin{array}{ccccc}
FY \# X & \xrightarrow{\text{id} \# d} & FY \# FY & \xrightarrow{\varphi_Y} & FY \\
\downarrow t \# e & & \downarrow t \# \text{id} & & \downarrow t \\
(Y \# FY) \# (FY \# X) & \xrightarrow{\text{id} \# (\text{id} \# d)} & (Y \# FY) \# (FY \# FY) & \xrightarrow{(Y \# \varphi_Y) \# \varphi_Y} & FY \\
\downarrow (\text{id} \# u_{FY}^X) \# \text{id} & & \uparrow \text{id} \# \varphi_Y & & \\
(Y \# (FY \# X)) \# (FY \# X) & \xrightarrow{(\text{id} \# (\text{id} \# d)) \# (\text{id} \# d)} & (Y \# (FY \# FY)) \# (FY \# FY) & \xrightarrow{m_Y^{FY} \# \text{id}} & Y \# FY \\
\downarrow m_Y^{FY \# X} & & \downarrow m_Y^{FY \# FY} & & \\
Y \# (FY \# X) & \xrightarrow{\text{id} \# (\text{id} \# d)} & Y \# (FY \# FY) & \xrightarrow{\text{id} \# \varphi_Y} & Y \# FY
\end{array}$$

Note first that the left-hand edge is  $\bar{e}$ . The upper left-hand square commutes since  $d$  is a solution of  $e$ , for the part below it use that  $(FY \# d)u_{FY}^X = u_{FY}^{FY}$  holds since  $FY \# d$  is a monad morphism, and the lower left-hand part commutes by the laws of  $\#$ . The upper right-hand part commutes by Corollary 37, and the remaining little inner triangle commutes since  $\varphi_Y u_{FY}^{FY} = \text{id}_{FY}$  since  $\varphi_Y$  is the structure of a  $\#$ -algebra. Hence  $\varphi_Y(FY \# d)$  is a coalgebra homomorphisms as desired, which completes the proof.  $\square$

### Proof of Lemma 24

To show the claim, form the following coalgebra for  $A \# -$ :

$$\begin{array}{c}
X \xrightarrow{e} B \# X \xrightarrow{f \# X} (A \# F_{\#} A) \# X \xrightarrow{(A \# \text{inl}) \# \text{inr}} (A \# (F_{\#} A + X)) \# (F_{\#} A + X) \\
\downarrow m_A^{F_{\#} A + X} \\
A \# (F_{\#} A + X)
\end{array}$$

By Proposition 23 we obtain a unique  $h : X \rightarrow F_{\#} A$  such that the diagram below commutes:

$$\begin{array}{ccc}
X & \xrightarrow{m_A^{F_{\#} A + X} (((\text{id} \# \text{inl}) f) \# \text{inr}) e} & A \# (F_{\#} A + X) \\
h \downarrow & & \downarrow \text{id} \# [\text{id}, h] \\
F_{\#} A & \xrightarrow{\text{out}} & A \# F_{\#} A.
\end{array}$$

Now use that  $\text{id} \# [\text{id}, h]$  is a monad morphism to see that, equivalently,  $h$  is unique such that (1) commutes.  $\square$

**Proof of Lemma 25**

Let  $g = (\text{out}^{-1} f)^* = \text{coit}(((\text{out}^{-1} f) \# \text{id}) \text{out}, \text{out})$ . Then we have

$$\begin{array}{ccccc}
 X & \xrightarrow{e} & B \# X & & \\
 \downarrow \text{coit}(e) & & \downarrow \text{id} \# \text{coit}(e) & & \\
 F_{\#} B & \xrightarrow{\text{out}} & B \# F_{\#} B & \xrightarrow{(\text{out}^{-1} f) \# \text{id}} & F_{\#} A \# F_{\#} A \\
 \downarrow g & & & & \downarrow m_A^{F_{\#} A} (\text{out} \# g) \\
 F_{\#} A & \xrightarrow{\text{out}} & A \# F_{\#} A & & 
 \end{array}$$

and the uppermost path from  $X$  to  $A \# F_{\#} A$  amounts to  $m_A^{F_{\#} A} (f \# (g \text{coit}(e)))$ . Therefore,  $g \text{coit}(e)$  satisfies the equation uniquely determining  $\text{coit}(e, f)$ , implying the result.  $\square$

**Proof of Lemma 26**

Notice that diagram (1) implies trivially that

$$\text{coit}(e, f) = \text{coit}((f \# \text{id}) e, \text{id}).$$

Thus we get

$$\begin{aligned}
 & (F_{\#} g) \text{coit}(e) \\
 & (\eta^{\nu} g)^* \text{coit}(e) \\
 & = (\text{out}^{-1} \text{out } \eta^{\nu} g)^* \text{coit}(e) \\
 & = \text{coit}(e, \text{out } \eta^{\nu} g) && // \text{ Lemma 25} \\
 & = \text{coit}(e, u g) \\
 & = \text{coit}((g \# \text{id}) e, u) && // \text{ definition of } \text{coit}(-, -) \\
 & = \text{coit}((g \# \text{id}) e). && // \text{ corollary to Lemma 25 } \quad \square
 \end{aligned}$$

Let introduce the following useful morphism:

$$\text{ext} = \text{out}^{-1}(\text{id} \# \eta^{\nu}) : X \# X \rightarrow F_{\#} X$$

natural in  $X$ .

**Proof of Theorem 27**

The proof is organized as follows. First we construct for each  $F_{\#}$ -algebra a complete Elgot  $\#$ -algebra and vice versa. Then we extend these constructions to functors and prove that these functors witness an isomorphism of categories.

Given an algebra  $\chi : F_{\#} A \rightarrow A$ , i.e.

$$\begin{array}{ccc}
 A & \xrightarrow{\eta^{\nu}} & F_{\#} A \\
 & \searrow \text{id} & \downarrow \chi \\
 & & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 F_{\#} F_{\#} A & \xrightarrow{F_{\#} \chi} & F_{\#} A \\
 \mu^{\nu} \downarrow & & \downarrow \chi \\
 F_{\#} A & \xrightarrow{\chi} & A
 \end{array}$$

we define a  $\#$ -algebra  $a : A \# A \rightarrow A$  as follows:

$$A \# A \xrightarrow{\text{ext}} F_{\#} A \xrightarrow{\chi} A$$

It is easy to see that  $a$  is an algebra for the monad  $- \# A$ . For any  $e : X \rightarrow A \# X$ , let  $e^{\dagger} : X \rightarrow A$  be given by  $\chi(\text{coit } e)$ .

We now need to check if the so-defined iteration operator satisfies the axioms of complete Elgot algebras.

**Solution.** To see that this holds, consider the following diagram, the outside of which constitutes the required property:

$$\begin{array}{ccccc}
 X & \xrightarrow{e} & A \# X & & \\
 \downarrow \text{coit } e & & \downarrow \text{id} \# \text{coit } e & & \\
 F_{\#} A & \xrightleftharpoons[\text{out}^{-1}]{\text{out}} & A \# F_{\#} A & & \\
 \downarrow \chi & & \downarrow \text{id} \# \chi & & \\
 A & \xleftarrow{a} & A \# A & \xleftarrow{e^{\dagger}} & X
 \end{array}$$

The top square obviously commutes as a finality diagram. For the lower square, we calculate

$$\begin{aligned}
 & \chi \text{ out}^{-1} \\
 &= \chi \mu^{\nu} \eta^{\nu} \text{ out}^{-1} \\
 &= \chi \mu^{\nu} \text{ out}^{-1} (\eta^{\nu} \# \eta^{\nu}) \\
 &= \chi F_{\#} \chi \text{ out}^{-1} (\eta^{\nu} \# \eta^{\nu}) \\
 &= \chi \text{ out}^{-1} (\chi \# F_{\#} \chi) (\eta^{\nu} \# \eta^{\nu}) \\
 &= \chi \text{ out}^{-1} (\text{id} \# \eta^{\nu} \chi) \\
 &= \chi \text{ out}^{-1} (\text{id} \# \eta^{\nu}) (\text{id} \# \chi) \\
 &= a(\text{id} \# \chi).
 \end{aligned}$$

**Functoriality.** This is a simple consequence of the definition of the dagger operation in terms of  $\text{coit}$ . Suppose that  $f h = (\text{id} \# h) e$ . Then

$$\begin{aligned}
 & \text{out}(\text{coit } f) h \\
 &= (\text{id} \# (\text{coit } f)) f h \\
 &= (\text{id} \# (\text{coit } f)) (\text{id} \# h) e \\
 &= (\text{id} \# ((\text{coit } f) h)) e,
 \end{aligned}$$

i.e.  $(\text{coit } f) h$  satisfies the identity uniquely characterizing  $\text{coit } e$ . Therefore  $(\text{coit } f) h = \text{coit}(e)$  and hence  $f^{\dagger} h = \chi(\text{coit } f) h = \chi(\text{coit } e) = e^{\dagger}$ .

**Compositionality.** We have on the one hand

$$(f^{\dagger} \bullet g)^{\dagger} = \chi \text{ coit}(f^{\dagger} \bullet g)$$

$$\begin{aligned}
&= \chi \text{ coit}((\chi(\text{coit } f) \# \text{id}) g) \\
&= \chi \text{ coit}((\chi \# \text{id})((\text{coit } f) \# \text{id}) g) \\
&= \chi (F_{\#} \chi) \text{ coit}(((\text{coit } f) \# \text{id}) g) && // \text{ Lemma 26} \\
&= \chi \mu^{\nu} \text{ coit}(((\text{coit } f) \# \text{id}) g) && // \chi \text{ is an } F_{\#}\text{-algebra} \\
&= \chi \text{ coit}(((\text{coit } f) \# \text{id}) g, \text{out}), && // \text{ Lemma 25}
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
(f \blacksquare g)^{\dagger} \text{ inr} &= \chi \text{ coit}(f \blacksquare g) \text{ inr} \\
&= \chi \text{ coit}(m_A^{Y+X} (((\text{id} \# \text{inl}) f) \# \text{inr}) [u_Y^X, g]) \text{ inr}.
\end{aligned}$$

Now, let  $h = m_A^{Y+X} (((\text{id} \# \text{inl}) f) \# \text{inr}) [u_Y^X, g]$ . We are finished once we proved that  $\text{coit}(h) \text{ inr}$  satisfies the identity characterizing  $\text{coit}((\text{coit}(f) \# \text{id}) g, \text{out})$ . First observe that the following:

$$\begin{aligned}
&\text{out}(\text{coit } h) \text{ inl} \\
&= (\text{id} \# \text{coit } h) h \text{ inl} \\
&= (\text{id} \# \text{coit } h) m_A^{Y+X} (((\text{id} \# \text{inl}) f) \# \text{inr}) u_Y^X \\
&= (\text{id} \# \text{coit } h) m_A^{Y+X} u_A^{Y+X} (\text{id} \# \text{inl}) f \\
&= (\text{id} \# \text{coit } h) (\text{id} \# \text{inl}) f \\
&= (\text{id} \# ((\text{coit } h) \text{ inl})) f,
\end{aligned}$$

i.e.  $(\text{coit } h) \text{ inl}$  satisfies the identity characterizing  $\text{coit}(f)$  and therefore

$$(\text{coit } h) \text{ inl} = \text{coit } f \quad (11)$$

Then we proceed as follows:

$$\begin{aligned}
&\text{out}(\text{coit } h) \text{ inr} \\
&= \text{out coit}(m_A^{Y+X} (((\text{id} \# \text{inl}) f) \# \text{inr}) [u_Y^X, g]) \text{ inr} \\
&= (\text{id} \# \text{coit } h) m_A^{Y+X} (((\text{id} \# \text{inl}) f) \# \text{inr}) g \\
&= m_A^{F_{\#} A} (((\text{id} \# ((\text{coit } h) \text{ inl})) f) \# ((\text{coit } h) \text{ inr})) g. \\
&= m_A^{F_{\#} A} (((\text{id} \# \text{coit } f) f) \# ((\text{coit } h) \text{ inr})) g && // (11) \\
&= m_A^{F_{\#} A} ((\text{out}(\text{coit } f)) \# ((\text{coit } h) \text{ inr})) g \\
&= m_A^{F_{\#} A} (\text{out} \# ((\text{coit } h) \text{ inr})) (\text{coit } f \# \text{id}) g
\end{aligned}$$

which, as indicated above, implies that  $(\text{coit } h) \text{ inr} = \text{coit}((\text{coit } f \# \text{id}) g, \text{out})$ .

We proceed with the converse construction: Given a complete Elgot  $\#$ -algebra  $a : A \# A \rightarrow A$ , we build a  $F_{\#}$ -algebra by iterating the structure of the final coalgebra  $\text{out} : F_{\#} A \rightarrow F_{\#} A \# A$ :

$$F_{\#} A \xrightarrow{\text{out}^{\dagger}} A$$

To show that  $\text{out}^\dagger$  is an  $F_\#$ -algebra, we check the following.

**Compatibility with unit.** Since  $\text{out } \eta^\nu = u_A^{F_\# A} = (A \# \eta^\nu) u_A^A$ , by the functoriality axiom,

$$\text{out}^\dagger \eta^\nu = (u_A^A)^\dagger.$$

Using the solution axiom, we obtain, since  $a$  is an  $(- \# A)$ -algebra,

$$\text{out}^\dagger \eta^\nu = (u_A^A)^\dagger = a(\text{id} \# (u_A^A)^\dagger) u_A^A = a u_A^A = \text{id}.$$

**Compatibility with multiplication.** We need to show that  $\text{out}^\dagger F_\#(\text{out}^\dagger) = \text{out}^\dagger \mu^\nu$ .

Note that the type of morphisms on the left and on the right hand sides is  $F_\# F_\# A \rightarrow A$ . We show that both morphisms are equal to  $(\text{out}^\dagger \bullet \text{out})^\dagger$  having the same type, which is itself by compositionality equal to  $(\text{out} \blacksquare \text{out})^\dagger \text{inr}$ .

For the left-hand side of the original equation we obtain this by functoriality:

$$\begin{aligned} \text{out } F_\#(\text{out}^\dagger) &= (\text{id} \# (F_\# \text{out}^\dagger)) (\text{out}^\dagger \# \text{id}) \text{out} \\ &= (\text{id} \# (F_\# \text{out}^\dagger)) (\text{out}^\dagger \bullet \text{out}) \end{aligned}$$

and therefore

$$\text{out}^\dagger F_\#(\text{out}^\dagger) = (\text{out}^\dagger \bullet \text{out})^\dagger.$$

As for the right-hand side, consider the following diagram:

$$\begin{array}{ccc} F_\# A + F_\# F_\# A & \xrightarrow{\text{out} \blacksquare \text{out}} & A \# (F_\# A + F_\# F_\# A) \\ \downarrow [\text{id}, \mu^\nu] & & \downarrow \text{id} \# [\text{id}, \mu^\nu] \\ F_\# A & \xrightarrow{\text{out}} & A \# F_\# A \end{array} \quad (12)$$

Let us verify that this diagram commutes by case distinction (we drop the indices at  $m$  and  $u$  for readability). On the one hand,  $\text{out} [\text{id}, \mu^\nu] \text{inl} = \text{out}$  and also

$$\begin{aligned} & (\text{id} \# [\text{id}, \mu^\nu]) (\text{out} \blacksquare \text{out}) \text{inl} \\ &= (\text{id} \# [\text{id}, \mu^\nu]) m (((\text{id} \# \text{inl}) \text{out}) \# \text{inr}) u \quad // \text{ definition of } \blacksquare \\ &= (\text{id} \# [\text{id}, \mu^\nu]) m u (\text{id} \# \text{inl}) \text{out} \\ &= (\text{id} \# [\text{id}, \mu^\nu]) (\text{id} \# \text{inl}) \text{out} \\ &= \text{out}; \end{aligned}$$

analogously, on the other hand,

$$\begin{aligned} & \text{out} [\text{id}, \mu^\nu] \text{inr} \\ &= \text{out } \mu^\nu \\ &= m (\text{out} \# \mu^\nu) \text{out} \end{aligned}$$

and

$$(\text{id} \# [\text{id}, \mu^\nu]) (\text{out} \blacksquare \text{out}) \text{inr}$$

$$\begin{aligned}
&= (\text{id} \# [\text{id}, \mu^\nu]) m (((\text{id} \# \text{inl}) \text{out}) \# \text{inr}) \text{out} \\
&= m ((\text{id} \# [\text{id}, \mu^\nu]) \# [\text{id}, \mu^\nu]) (((\text{id} \# \text{inl}) \text{out}) \# \text{inr}) \text{out} \\
&= m ((\text{id} \# [\text{id}, \mu^\nu] \text{inl}) \text{out} \# [\text{id}, \mu^\nu] \text{inr}) \text{out} \\
&= m (\text{out} \# \mu^\nu) \text{out}.
\end{aligned}$$

From (12), by functoriality, we obtain

$$\text{out}^\dagger [\text{id}, \mu^\nu] = (\text{out} \blacktriangle \text{out})^\dagger$$

and thus

$$\text{out}^\dagger \mu^\nu = (\text{out} \blacktriangle \text{out})^\dagger \text{inr}.$$

Let us now complete the constructed correspondence between  $|\mathbf{C}^{F\#}|$  and  $|\mathbf{CElg}_\#(\mathbf{C})|$  to an equivalence of categories. Let  $F : \mathbf{C}^{F\#} \rightarrow \mathbf{CElg}_\#(\mathbf{C})$  be defined as follows:  $F$  assigns to an  $F_\#$ -algebra  $(A, \chi)$  the complete Elgot algebra  $(A, \chi \text{ ext}, -^\dagger)$  with the iteration as presented above, and to an  $F_\#$ -algebra homomorphism  $f : (A, \chi) \rightarrow (B, \zeta)$  the underlying morphism from  $A$  to  $B$ . Let us check that this definition is correct, i.e. the above  $f$  is a complete Elgot  $\#$ -algebra morphism from  $F(A, \chi) = (A, a, -^\dagger)$  to  $F(B, \zeta) = (B, b, -^\dagger)$ , i.e. for any  $e : X \rightarrow A \# X$ :

$$\begin{aligned}
f e^\dagger &= f \chi (\text{coit } e) && // \text{definition of } -^\dagger \\
&= \zeta (F_\# f) (\text{coit } e) && // f \text{ is a } F_\# \text{-algebra morphism} \\
&= \zeta \text{coit} ((f \# \text{id}) e) && // \text{Lemma 26} \\
&= ((f \# \text{id}) e)^\dagger \\
&= (f \bullet e)^\dagger.
\end{aligned}$$

For the converse direction, let  $G : \mathbf{CElg}_\#(\mathbf{C}) \rightarrow \mathbf{C}^{F\#}$  send a complete Elgot  $\#$ -algebra  $(A, a, \dagger)$  to  $(A, \text{out}^\dagger)$ , which we proved to be an  $F_\#$ -algebra. Given a morphism  $f : (A, a, \dagger) \rightarrow (B, b, \dagger)$ , let  $Gf = f$  and let us show that  $f$  is indeed an  $F_\#$ -algebra morphism from  $(A, \text{out}^\dagger)$  to  $(B, \text{out}^\dagger)$ . By functoriality,

$$\text{out} (F_\# f) = (\text{id} \# F_\# f) (f \# \text{id}) \text{out}$$

implies

$$\text{out}^\dagger (F_\# f) = ((f \# \text{id}) \text{out})^\dagger = (f \bullet \text{out})^\dagger.$$

But, by definition of complete Elgot  $\#$ -algebra morphisms,

$$(f \bullet \text{out})^\dagger = f \text{out}^\dagger,$$

so the functor  $G$  is well-defined.

To finish the proof, we need to show that both  $GF$  and  $FG$  are identities. Since both functors act as the identity on morphisms, we only need to verify this on objects. On the one hand,

$$(GF)(A, \chi) = G(A, \chi \text{ ext}, -^\dagger) = (A, \text{out}^\dagger) = (A, \chi),$$



for  $\text{out}^\dagger$  is defined as  $\chi \text{ coit}(\text{out})$  and  $\text{coit}(\text{out})$  is the identity. Similarly,

$$(FG)(A, a, -^\dagger) = F(A, \text{out}^\dagger) = (A, \text{out}^\dagger \text{ ext}, -^\dagger) = (A, a, -^\dagger),$$

since, by functoriality applied to  $\text{out} \text{ coit}(e) = (\text{id} \# \text{coit}(e)) e$ ,

$$e^\dagger = \text{out}^\dagger \text{ coit}(e) = e^\dagger,$$

whatever  $e : X \rightarrow A \# X$  is, and moreover

$$\begin{aligned} \text{out}^\dagger \text{ ext} &= a (\text{id} \# \text{out}^\dagger) \text{ out ext} && // \text{ solution} \\ &= a (\text{id} \# \text{out}^\dagger) (\text{id} \# \eta^\nu) && // \text{ definition of ext} \\ &= a. && // \text{ out}^\dagger \text{ is an } F_\# \text{-algebra} \end{aligned}$$

□

### Proof of Proposition 31

In this proof all morphisms and compositions are in the Kleisli category of the complete Elgot monad  $\mathbb{T}$ , and we denote identity morphisms by their (co)domain. Note that the codiagonal law can, equivalently, be written as

$$((Y + \nabla)e)^\dagger = e^{\dagger\dagger}$$

for any  $e : X \rightarrow Y + X + X$ , where  $\nabla = [\text{inl}, \text{inr}]$  is the codiagonal (hence the name of the law).

Let  $g : X \rightarrow Y + X$  and  $f : Y \rightarrow Z + Y$  and form the following morphism

$$w = (Y + X \xrightarrow{[\text{inl}, g]} Y + X \xrightarrow{f+X} Z + Y + X \xrightarrow{Z+\text{inl}+\text{inr}} Z + (Y + X) + (Y + X)).$$

Now observe that the left-hand morphism of (4) is  $((Z + \nabla)w)^\dagger \text{ inr}$ . By the codiagonal law we have

$$((Z + \nabla)w)^\dagger \text{ inr} = w^{\dagger\dagger} \text{ inr}.$$

So it remains to prove that  $w^{\dagger\dagger} \text{ inr} = f^\dagger g^\dagger$ . Clearly, we have

$$w = ((Z + \text{inl})f + (Y + X))(Y + \text{inr})[\text{inl}, g]. \quad (13)$$

Since  $((Y + \text{inr})[\text{inl}, g]) \text{ inr} = (Y + \text{inr})g$  we obtain by functoriality that

$$((Y + \text{inr})[\text{inl}, g])^\dagger \text{ inr} = g^\dagger. \quad (14)$$

Now we compute

$$\begin{aligned} w^\dagger &= (Z + \text{inl})f((Y + \text{inr})[\text{inl}, g])^\dagger && // \text{ (13) and naturality} \\ &= (Z + \text{inl})f[Y, ((Y + \text{inr})[\text{inl}, g])^\dagger](Y + \text{inr})[\text{inl}, g] && // \text{ fixpoint} \end{aligned}$$

$$\begin{aligned}
&= (Z + \text{inl})f[Y, ((Y + \text{inr})[\text{inl}, g])^\dagger \text{inr}][\text{inl}, g] \\
&= (Z + \text{inl})f[Y, g^\dagger][\text{inl}, g] && // (14) \\
&= (Z + \text{inl})f[[Y, g^\dagger] \text{inl}, [Y, g^\dagger]g] \\
&= (Z + \text{inl})f[Y, g^\dagger] && // \text{fixpoint}
\end{aligned}$$

Now observe that  $(Z + \text{inl})f[Y, g^\dagger] \text{inl} = (Z + \text{inl})f$  so that functoriality gives us

$$((Z + \text{inl})f[Y, g^\dagger])^\dagger \text{inl} = f^\dagger. \quad (15)$$

Finally, we compute

$$\begin{aligned}
w^{\dagger\dagger} \text{inr} &= ((Z + \text{inl})f[Y, g^\dagger])^\dagger \text{inr} \\
&= [Z, ((Z + \text{inl})f[Y, g^\dagger])^\dagger](Z + \text{inl})f[Y, g^\dagger] \text{inr} && // \text{fixpoint} \\
&= [Z, ((Z + \text{inl})f[Y, g^\dagger])^\dagger \text{inl}]fg^\dagger \\
&= [Z, f^\dagger]fg^\dagger && // (15) \\
&= f^\dagger g^\dagger && // \text{fixpoint}
\end{aligned}$$

This completes the proof.  $\square$

### Proof of Theorem 32

First note that (4) can, equivalently, be rewritten as

$$((\eta \oplus \text{inl})f, \text{inr inr}) \diamond [\text{inl}, g]^\dagger \text{inr} = f^\dagger \diamond g^\dagger \quad (16)$$

where  $g : X \multimap Y + X$ ,  $f : Y \multimap Z + Y$ .

(i) Let us prove the first clause. By Corollary 13,  $\mathbf{C}^\mathbb{T}$  is isomorphic to the category of all those  $\#$ -algebras whose structure factor through  $T\nabla$ , specifically, every  $\mathbb{T}$ -algebra  $(A, a : TA \rightarrow A)$  gives rise to a  $\#$ -algebra  $(A, a(T\nabla) : T(A + A) \rightarrow A)$ . In the case at hand, we equip every such  $(A, a(T\nabla) : T(A + A) \rightarrow A)$  with an iteration  $-^\dagger$  operator sending any  $e : X \rightarrow T(A + X)$  to  $e^\dagger = ae^\dagger : X \rightarrow A$ .

Let us check the axioms of complete Elgot  $\#$ -algebras.

– *Solution.* This follows easily from the *fixpoint* property of the dagger of the complete Elgot monad  $\mathbb{T}$ :

$$\begin{aligned}
e^\dagger &= ae^\dagger \\
&= a[\eta, e^\dagger]^\star e \\
&= a\mu T[\eta, e^\dagger] e \\
&= a(Ta)T[\eta, e^\dagger] e \\
&= aT[\text{id}, e^\dagger] e \\
&= a(T\nabla)T(\text{id} + e^\dagger) e \\
&= a(T\nabla)(\text{id} \# e^\dagger) e.
\end{aligned}$$

- *Functoriality* is a trivial application of *uniformity*:

$$f h = (\text{id} \# h) e = T(\text{id} + h) e$$

implies

$$f^\dagger h = a f^\dagger h = a e^\dagger = e^\dagger.$$

- *Compositionality*. Since  $X \# Y = T(X + Y)$ , we have

$$f \blacksquare g = [T(\text{id} + \text{inl}) f, \eta \text{ inr inr}]^* [\eta \text{ inl}, g] : Y + X \longrightarrow T(A + (Y + X)).$$

Hence, by (16),

$$(f \blacksquare g)^\dagger \text{ inr} = f^\dagger \diamond g^\dagger.$$

Composing with  $a : TA \rightarrow A$  we obtain

$$(f \blacksquare g)^\dagger \text{ inr} = a(f^\dagger \diamond g^\dagger).$$

Let us further rewrite the right hand side.

$$\begin{aligned} a(f^\dagger \diamond g^\dagger) &= a \mu T(f^\dagger) g^\dagger \\ &= a(Ta) T(f^\dagger) g^\dagger \\ &= a T(a f^\dagger) g^\dagger \\ &= a(T(a f^\dagger + \text{id}) g)^\dagger && // \text{ naturality} \\ &= ((f^\dagger \# \text{id}) g)^\dagger \\ &= (f^\dagger \bullet g)^\dagger. \end{aligned}$$

Next, let us show that any  $\mathbb{T}$ -algebra morphism  $h : A \rightarrow B$  from  $(A, a)$  to  $(B, b)$  gives rise to a complete Elgot  $\#$ -algebra morphism between the corresponding  $\#$ -algebras, i.e. for every  $f : X \rightarrow A \# X$  we have

$$h f^\dagger = ((h \# \text{id}) f)^\dagger.$$

Indeed,

$$\begin{aligned} h f^\dagger &= h a f^\dagger \\ &= b(Th) f^\dagger \\ &= b(T(h + \text{id}) f)^\dagger && // \text{ naturality} \\ &= ((h \# \text{id}) f)^\dagger. \end{aligned}$$

We have constructed a functor from  $\mathbf{C}^\mathbb{T}$  to  $\mathbf{CElg}_\#(\mathbf{C})$ . This functor is full and faithful because the underlying functor from  $\mathbf{C}^\mathbb{T}$  into the category of  $\#$ -algebras is full and faithful by Corollary 13 and any morphism of complete  $\#$ -algebras is a morphism of  $\#$ -algebras by Proposition 18.

Finally, let us check that any complete Elgot  $\#$ -algebra of the form  $(A, a(T\nabla), -^\dagger)$  satisfying  $e^\dagger = a(T\nabla)(T\text{inl})e^\dagger$  for every  $e : X \rightarrow T(A + X)$ ,

is an image of a  $\mathbb{T}$ -algebra, specifically of  $(A, a)$ . We only have to verify that  $(A, a)$  is indeed a  $\mathbb{T}$ -algebra. This is however straightforward from the axioms of  $\#$ -algebras and the definitions  $u_X^Y = \eta \text{ inl}$  and  $m_X^Y = [\text{id}, \eta \text{ inr}]^*$ .

(ii) We now proceed with the second clause. To that end we have to verify the axioms of weak complete Elgot monads.

– *Fixpoint.* Given  $f : X \rightarrow T(Y + X)$ ,

$$\begin{aligned} f^\dagger &= (T(\eta + \text{id}) f)^\ddagger \\ &= \mu (T\nabla) T(\text{id} + f^\dagger) T(\eta + \text{id}) f && // \text{ solution} \\ &= [\eta, f^\dagger]^* f. \end{aligned}$$

– *Naturality.* Let  $f : X \rightarrow T(Y + X)$  and let  $g : Y \rightarrow TZ$ . We consider two special cases  $g = \text{id} : TY \rightarrow TY$  and  $g = \eta h$ , where  $h : Y \rightarrow Z$ , i.e. we will prove

$$(\eta h)^* f^\dagger = ([\eta \text{ inl } h, \eta \text{ inr}]^* f)^\dagger \quad (17)$$

$$\text{id}^* f^\dagger = ([T \text{ inl}, \eta \text{ inr}]^* f)^\dagger, \quad (18)$$

which jointly imply the general case as follows:

$$\begin{aligned} g \diamond f^\dagger &= g^* f^\dagger \\ &= \mu (Tg) f^\dagger \\ &= \mu (\eta g)^* f^\dagger \\ &= \mu ([\eta \text{ inl } g, \eta \text{ inr}]^* f)^\dagger && // (17) \\ &= \text{id}^* ([\eta \text{ inl } g, \eta \text{ inr}]^* f)^\dagger \\ &= ([T \text{ inl}, \eta \text{ inr}]^* [\eta \text{ inl } g, \eta \text{ inr}]^* f)^\dagger && // (18) \\ &= ([ (T \text{ inl}) g, \eta \text{ inr}]^* f)^\dagger \\ &= ((g \oplus \eta) \diamond f)^\dagger. \end{aligned}$$

The proof of (17) is based on the fact that  $Th : TY \rightarrow TZ$  is a morphism of  $\mathbb{T}$ -algebras from  $(TY, \mu)$  to  $(TZ, \mu)$  and hence, by assumption,  $h$  is a morphism of  $\#$ -algebras from  $J(TY, \mu)$  to  $J(TZ, \mu)$ :

$$\begin{aligned} (\eta h) \diamond f^\dagger &= (\eta h)^* f^\dagger \\ &= (Th) (T(\eta + \text{id}) f)^\ddagger \\ &= (T(Th + \text{id}) T(\eta + \text{id}) f)^\ddagger \\ &= (T(\eta h + \text{id}) f)^\ddagger \\ &= (T(\eta + \text{id}) [\eta \text{ inl } h, \eta \text{ inr}]^* f)^\ddagger \\ &= ([\eta \text{ inl } (\eta h), \eta \text{ inr}]^* f)^\ddagger \\ &= ((\eta h) \oplus \eta) \diamond f)^\ddagger. \end{aligned}$$

Now we prove (18). In this case we have  $f : X \rightarrow T(TY + X)$ . We apply compositonality to  $f$  and  $T(\text{inl } \eta) : TY \rightarrow T(TY + TY)$  to obtain

$$(T(\text{inl } \eta) \blacksquare f)^\dagger \text{ inr} = ((T(\text{inl } \eta))^\dagger \bullet f)^\dagger. \quad (19)$$

First of all, note that

$$\begin{aligned} (T(\text{inl } \eta))^\dagger &= [\text{id}, \eta \text{ inr}]^* T(\text{id} + (T(\text{inl } \eta))^\dagger) T(\text{inl } \eta) && // \text{ fixpoint} \\ &= [\text{id}, \eta \text{ inr} (T(\text{inl } \eta))^\dagger]^* T(\text{inl } \eta) \\ &= \eta^* \\ &= \text{id}. \end{aligned}$$

Using the fact that  $\text{id}^* = \mu$  is a morphism of complete Elgot #-algebras, we obtain the left hand side of (18) from the left-hand side of (19)

$$\begin{aligned} ((T(\text{inl } \eta))^\dagger \bullet f)^\dagger &= (T((T(\text{inl } \eta))^\dagger + \text{id}) f)^\dagger \\ &= f^\dagger \\ &= (T(\mu + \text{id}) T(\eta + \text{id})) f^\dagger \\ &= \mu (T(\eta + \text{id}) f)^\dagger \\ &= \text{id}^* f^\dagger. \end{aligned}$$

In order to prove that we obtain the right-hand side of (18) from the right-hand side of (19), let us denote  $T(\text{inl } \eta) \blacksquare f : TY + X \rightarrow T(TY + (TY + X))$  by  $t$ . Then, by definition,

$$\begin{aligned} t &= m_{TY+X}^{TY+X} (((\text{id} \# \text{inl}) T(\text{inl } \eta)) \# \text{inr}) [u_{TY}^X, f] \\ &= \mu T[\text{id}, \eta \text{ inr}] \underbrace{T(T(\text{id} + \text{inl}) T(\text{inl } \eta) + \text{inr})}_{T(\text{inl } \eta)} [\eta \text{ inl}, f] \\ &= \mu T[T(\text{inl } \eta), \eta \text{ inr inr}] [\eta \text{ inl}, f] \\ &= [T(\text{inl } \eta), \eta \text{ inr inr}]^* [\eta \text{ inl}, f] \\ &= [T(\text{inl } \eta), [T(\text{inl } \eta), \eta \text{ inr inr}]^* f]. \end{aligned}$$

Observe that we have

$$\begin{aligned} t \text{ inr} &= [T(\text{inl } \eta), \eta \text{ inr inr}]^* f \\ &= T(\text{id} + \text{inr}) [T(\text{inl } \eta), \eta \text{ inr}]^* f \\ &= (\text{id} \# \text{inr}) [T(\text{inl } \eta), \eta \text{ inr}]^* f. \end{aligned}$$

Therefore, using uniformity we obtain

$$\begin{aligned} t^\dagger \text{ inr} &= ([T(\text{inl } \eta), \eta \text{ inr}]^* f)^\dagger \\ &= (T(\eta + \text{id}) [T \text{ inl}, \eta \text{ inr}]^* f)^\dagger \\ &= ([T \text{ inl}, \eta \text{ inr}]^* f)^\dagger, \end{aligned}$$

which is the right hand side of (18).

– *Uniformity.* Suppose,  $f : X \rightarrow T(Y + X)$ ,  $g : Z \rightarrow T(Y + Z)$ ,  $h : Z \rightarrow X$  and  $f h = T(\text{id} + h) g$ . The latter implies  $T(\eta + \text{id}) f h = T(\eta + \text{id}) T(\text{id} + h) g = T(\text{id} + h) T(\eta + \text{id}) g$  and hence we can apply functoriality of  $-^\dagger$ :

$$\begin{aligned} f^\dagger h &= (T(\eta + \text{id}) f)^\dagger h \\ &= (T(\eta + \text{id}) g)^\dagger && // \text{ functoriality} \\ &= g^\dagger \end{aligned}$$

as required.

Finally, let us check (16). We start with the following instance of compositionality:

$$([T(\eta + \text{inl}) f, \eta \text{inr inr}]^\star [\eta \text{inl}, g])^\dagger \text{inr} = ((T(\eta + \text{id}) f \blacksquare g)^\dagger = (T((T(\eta + \text{id}) f)^\dagger + \text{id})) g)^\dagger$$

where we used the assumption that  $J(TZ, \mu)$  is a complete Elgot  $\#$ -algebra. Now,

$$\begin{aligned} &([T(\eta + \text{inl}) f, \eta \text{inr inr}]^\star [\eta \text{inl}, g])^\dagger \text{inr} \\ &= (T(\eta + \text{id}) [T(\text{id} + \text{inl}) f, \eta \text{inr inr}]^\star [\eta \text{inl}, g])^\dagger \text{inr} \\ &= ([T(\text{id} + \text{inl}) f, \eta \text{inr inr}]^\star [\eta \text{inl}, g])^\dagger \text{inr} \\ &= ((\eta \oplus \underline{\text{inl}}) f, \underline{\text{inr inr}}] \diamond [\underline{\text{inl}}, g])^\dagger \text{inr} \\ &= (T((T(\eta + \text{id}) f)^\dagger + \text{id})) g)^\dagger \\ &= (T(f^\dagger + \text{id}) g)^\dagger \\ &= (T(\mu + \text{id}) T((\eta T) + \text{id}) T(f^\dagger + \text{id}) g)^\dagger \\ &= \mu (T((\eta T) + \text{id}) T(f^\dagger + \text{id}) g)^\dagger && // \mu \text{ is a morphism in } \mathbf{CElg}_\#(\mathbf{C}) \\ &= \mu (T(f^\dagger + \text{id}) g)^\dagger \\ &= \mu (T f^\dagger) g^\dagger && // \text{ naturality of } -^\dagger \\ &= f^\dagger \diamond g^\dagger \end{aligned}$$

which in summary yields (16).  $\square$

### Proof of Theorem 33

Suppose that  $\mathbb{T}$  is a complete Elgot monad and let us show (6). Note the identity

$$f^\ddagger = a f^\dagger, \tag{20}$$

which holds for every  $f : X \rightarrow A \# X$  and every  $\mathbb{T}$ -algebra  $(A, a)$  because  $a : TA \rightarrow A$  is a morphism of  $\mathbb{T}$ -algebras from  $(TA, \mu)$  to  $(A, a)$ , hence a morphism of complete Elgot  $\#$ -algebras from  $J(TA, \mu)$  to  $J(A, a)$  and therefore we have

$$f^\ddagger = ((a \# \text{id}) (\eta \# \text{id}) f)^\dagger = a ((\eta \# \text{id}) f)^\dagger = a f^\dagger.$$

Now, (6) is obtained as follows: for any  $e : X \rightarrow (A \# X) \# X = T(T(A + X) + X)$  we have

$$\begin{aligned}
(m_{A \# X}^X e)^\dagger &= ([\text{id}, \eta \text{inr}]^\star e)^\dagger \\
&= (T(a + \text{id}) [T(\eta + \text{id}), \eta \text{inr}]^\star e)^\dagger \\
&= a ([T(\eta + \text{id}), \eta \text{inr}]^\star e)^\dagger && // \text{ } a \text{ preserves } -^\dagger \\
&= a (T(\eta + \text{id}) [\text{id}, \eta \text{inr}]^\star e)^\dagger \\
&= a ([\text{id}, \eta \text{inr}]^\star e)^\dagger && // \text{ (5)} \\
&= a (T[\text{id}, \text{inr}] [T \text{inl}, \eta \text{inr}]^\star e)^\dagger \\
&= a (([T \text{inl}, \eta \text{inr}]^\star e)^\dagger)^\dagger && // \text{ codiagonal} \\
&= a (((\text{id} \oplus \eta) \diamond e)^\dagger)^\dagger \\
&= a (\text{id}^\star e^\dagger)^\dagger && // \text{ naturality} \\
&= a (\mu e^\dagger)^\dagger \\
&= (e^\dagger)^\dagger. && // \text{ (20)}
\end{aligned}$$

Conversely, we assume (6) and prove that  $\mathbb{T}$  is a complete Elgot monad. By Theorem 32, we only need to verify the codiagonal identity. Let  $f : X \rightarrow T((Y + X) + X)$  and let us take

$$e = T(\eta(\eta + \text{id}) + \text{id}) f : X \rightarrow T(T(TY + X) + X) = (TY \# X) \# X$$

in (6). Then we obtain the codiagonal identity for  $f$  as follows:

$$\begin{aligned}
(T[\text{id}, \text{inr}] f)^\dagger &= (T(\eta + \text{id}) T[\text{id}, \text{inr}] f)^\dagger \\
&= (T[\eta + \text{id}, \text{inr}] f)^\dagger \\
&= ([\eta(\eta + \text{id}), \eta \text{inr}]^\star f)^\dagger \\
&= ([\text{id}, \eta \text{inr}]^\star T(\eta(\eta + \text{id}) + \text{id}) f)^\dagger \\
&= ((T(\eta(\eta + \text{id}) + \text{id}) f)^\dagger)^\dagger && // \text{ (6)} \\
&= ((T(\eta + \text{id}) T((\eta + \text{id}) + \text{id}) f)^\dagger)^\dagger \\
&= ((T((\eta + \text{id}) + \text{id}) f)^\dagger)^\dagger \\
&= (T(\eta + \text{id}) f^\dagger)^\dagger && // \text{ naturality} \\
&= (f^\dagger)^\dagger.
\end{aligned}$$

This completes the proof.  $\square$